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STRESSES IN MASONRY

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PREFACE

This little work supplies, I believe, a felt want. Elementary treatises on steel-work are numerous, but a brief yet comprehensive account of the stresses in masonry has not yet, to my knowledge, been produced.

I have assumed that the reader is familiar with mathematics only up to quadratics, all more advanced matter being relegated to foot-notes. Also I have assumed that a knowledge of stresses in steel, such as is possessed by the majority of young engineers and architects in practice, is at the reader's disposal.

Beyond this, I have endeavoured to make the matter as simple as possible, eliminating complex theory which may be a little more accurate, but does not adapt itself to design.

One feature in special I wish to emphasise: the use of twice the safe shearing stress as the safe compressive

stress instead of the direct safe compressive stress. The reasons for this will be apparent after reading Chapter I., and, at the same time, the common but false notion that unsafe stresses are uncommon in masonry will be avoided.

H. CHATLEY.

SOUTHSEA,

1909.

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REFERENCES

Perry's *Applied Mechanics*
Ewing's *Strength of Materials*
Rankine's *Civil Engineering*
Unwin's *Testing of Materials*

STRESSES IN MASONRY

CHAPTER I

STRENGTH OF STONE

In most cases masonry is subjected to only two kinds of stress—compression and shearing. Tension occurs but rarely, and then generally as the result of bending rather than of direct pulling.

As will be shown immediately, it is the shearing stress which is generally responsible for failure rather than compression, but the latter is most usually taken as the basis of strength-measurement.

Before proceeding, it will therefore be as well to have some simple figures to refer to. In each case a low value is given for the ultimate stress, and the stone is assumed to be of good average quality.

Ultimate Stresses (tons per sq. ft.)

Name.	Crushing.	Tension.	Shearing.	Bending.
Granite . . .	900	30	50	100
Basalt . . .	800	80	40	—

Name.	Crushing.	Tension.	Shearing.	Bending.
Slate . . .	800	60	40	—
Sandstone . .	500	10	30	50
Sandstone (soft) .	200	5	10	20
Marble . .	600	30	50	—
Limestone . .	500	25	40	60
Limestone (soft) .	100	9	35	50
Chalk . . .	10	—	—	—

These figures are selected and simplified from Molesworth's *Note-book* and Unwin's *Testing of Materials.* The majority are on the basis of Bauschinger's research.*

We shall refer to this table from time to time as a basis for calculations. It is useless employing very exact figures, since every specimen, even from the same quarry, will vary somewhat in strength.

When a column of masonry is in compression, it is more likely to fail by shearing obliquely than by direct compression, because the shearing stress at which breaking occurs is lower than the compressive ultimate stress.

Thus on an oblique section inclined θ to the horizontal, we have a normal and a parallel force. The

* *Mittheilungen aus dem Mech. Tech. Laboratorium in München* 1874.

latter $= P \sin\theta$. The area, if the column is square, is $D^2 \sec\theta$, so that the shearing stress is $\frac{P \sin\theta}{D^2 \sec\theta}$.

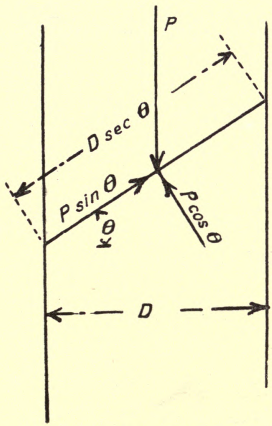

FIG. 1.

This expression is a maximum when θ is 45°, so that usually we find the column fractures into pyramidal pieces, the angle of slope being 45°.*

For this angle, $\sin\theta / \sec\theta = \frac{1}{2}$, so that the maximum shearing force per unit area is half the compressive

$$* \ \frac{d}{d\theta}\left(\frac{\sin\theta}{\sec\theta}\right) = \frac{\sec\theta \times \cos\theta - \sec\theta \times \tan\theta \times \sin\theta}{\sec^2\theta} = 0.$$

Hence $\sec\theta \times \cos\theta - \sec\theta \times \tan\theta \times \sin\theta = 0$

$$\cos\theta = \tan\theta$$

$$\tan^2\theta = \cos^2\theta = 1$$

$$\tan\theta = \cos\theta = \sqrt{1} = 1, \quad \theta = 45°.$$

stress. It is probable that this value is not fully reached, since there is a frictional resistance as well to be considered; but since the ultimate crushing stress is generally twenty or more times the shearing stress, it is obvious that the material will generally fail in the latter manner.

This is the most general case of fracture in masonry, and serves to explain the diagonal cracks which appear in faulty or decaying work.

Many masonry structures, such as retaining walls, tall chimneys, and to a certain extent arches, are liable to tensile stress; but it is a standard principle to avoid this whenever possible, particularly when special precautions have not been taken to ensure soundness in the joints. In fact, most of the theory of stress in masonry is based on the assumption of uncemented joints—weight, fit, and friction alone being relied on.

Stone is one of the least elastic substances, the value of the modulus of elasticity $\left(\frac{\text{stress per unit area}}{\text{strain per unit length}}\right)$ being from 20,000 tons per sq. ft. (Bunter Sandstone) to 685,000 tons per sq. ft. (Nummulitic Limestone); * or

* This may be compared with steel as follows:

Modulus of stone, 2 to 13 million lbs. per sq. in.

Modulus of steel, 30 million lbs. per sq. in., or about 1,872,000 tons per sq. ft.

taking 144,000 tons as a convenient and fair average, we have

$$\text{strain} = \frac{\text{stress}}{144{,}000},$$

or for a stress of 144 tons per sq. ft. (1 ton per sq. in.) the strain is $\frac{1}{1000}$. As will be seen from the table given above, in tension no stone can bear this stress, although in compression it will do so, apparently. It will not actually bear it, for the concurrent shearing stress along any diagonal plane inclined 45° to the horizontal is upwards of $144 \div 2 = 72$ tons per sq. ft., which is above the figures given. As a practical rule it may therefore be said that scarcely any stone will bear an extension or compression amounting to $\frac{1}{1000}$ of its linear dimensions. Since for ordinary blocks of stone this elongation or shortening is imperceptible, we arrive at the common result that stone is not visibly elastic.

A further question of some importance in connection with arches and retaining walls is that of friction between stone blocks. Although only in rare cases are blocks laid uncemented in a large structure, it is, as has been mentioned, usual to assume that the only resistance existing in the joints to shearing is that arising from the friction between the blocks.

According to General Morin, the value of μ, the coefficient of friction between stones, is 0·71 (*i.e.*, the

angle of friction is about $35\frac{1}{2}°$). It must, however, be recognised that this value will depend on the manner in which the surface of the stone is finished, and also to some extent on the nature of the stone. We may take it that this refers to sawn faces ot a moderately coarse stone. An example of the application of this to the shearing of piers already referred to may be given.

On the diagonal plane there is a normal pressure P cos θ or P $\cos^2 \theta \div D^2$ per sq. ft. If we multiply this by μ, we have the theoretical frictional resistance to motion on this plane, so that we may write

$$\frac{S}{D^2 \sec \theta} = \frac{P \sin \theta}{D^2 \sec \theta} - \frac{\mu P \cos \theta}{D^2 \sec \theta},$$

where S is the total shearing force.

This simplifies to

$$\begin{aligned} S &= P \sin \theta - \mu P \cos \theta \\ &= P (\sin \theta - \mu \cos \theta) \qquad (1) \end{aligned}$$

where S is the shearing force per unit area and μ P is the compressive stress per unit area.

If $\theta = 45°$, then we have

$$S = \cdot 2052 \text{ P} \qquad (1a)$$

Since the shearing force at fracture is generally less than $0{\cdot}1 \times$ the compression force, the probability is still in favour of the failure taking place by shearing.

Referring again to arches, it will be seen that it is possible to set voussoirs until the angle made by the joint with the horizontal is upwards of 71°. It must, however, be pointed out that, if the joints are thick or the cement at all liquid, the angle of friction will probably be much less and it will *not* be possible to set the voussoirs up to this angle.

Yet another application of the law of friction suggests itself in regard to the slipping of blocks along their bed joints. If a lateral force be applied which exceeds $\mu \times$ the weight on the joint, the block will slip unless dowelled.

We must now devote a little attention to the question of the strength of joints; for although this does not usually enter into calculations, it must necessarily do so in some special cases. Moreover, it will be useful to know what surplus of strength we have in ordinary cases.

The cementing materials employed are generally—

1. *Hydraulic lime mortar.*
2. *Portland cement.*
3. *Portland cement mortar.*

It is usual to determine the strength of these materials by tensile tests, although they are nearly always used in compressive stress. On account of these

materials hardening with age it is convenient to use Unwin's formula—

$$y = a + b\sqrt[3]{x - 1},$$

where y is the tensile strength at any time, x weeks after making, a is the strength one week after making, and b is a constant.

The following results obtained by Mr. Elliott Clarke are typical: *

Neat cement . .	$y = 303 + 61\sqrt[3]{x-1}$
1 to 1 . . .	$y = 160 + 57\sqrt[3]{x-1}$
1 to 2 . . .	$y = 126 + 44\sqrt[3]{x-1}$
1 to 3 . . .	$y = 95 + 36\sqrt[3]{x-1}$
1 to 5 . . .	$y = 55 + 26\sqrt[3]{x-1}$

The compressive strength of neat cement is about 2 tons per sq. in., *i.e.*, about 300 tons per sq. ft. Of *lime* mortar the compressive strength varies from 10 to 40 tons per sq. ft. According to Bauschinger, the ratio $\frac{\text{crushing strength}}{\text{tensile strength}}$ for neat cement varies from 7 to 11, 10 being usually taken.

The modulus of elasticity of cement is about 250,000 tons per sq. ft. As regards the shearing strength there seems to be considerable doubt, but 75 lbs. per sq. in. is a commonly assumed figure.

* Unwin's *Testing of Materials.*

Obviously the strength of masonry depends to a great extent on the strength of joints, particularly in regard to shearing. As to direct compression, the importance of the joint varies with the nature of the work. Thus in random rubble the mortar is all-important. In heavy masonry the mortar is scarcely of any importance.

In the case of random rubble, assuming the above figure for shearing, we have

$$75 \times 2 \times 144 = \text{say } 10 \text{ tons}$$

per sq. ft. as the crushing load. This is probably the minimum for any variety of masonry.

For granite, on the other hand, we have

$$50 \times 2 = 100 \text{ tons}$$

per sq. ft. as the crushing load which produces failure by shearing. All other cases will probably lie between these two, and the next step is to deduce from the crushing load a safe working load.

This is of course done by use of a " factor of safety," but, unfortunately, factors of safety are more uncertain than they are in steel work, where they are sufficiently so. Stone may contain unseen but dangerous flaws, and we are dependent on joints of uncertain workmanship made with material of greatly varying strength. Furthermore, masonry is incapable of retaining much strain energy, so that shocks are liable to cause fracture

which would not occur under similar circumstances with metal. Hence it is very usual to employ factors of safety varying from 7 to 12. 10 is a commonly accepted value.

Hence the working values for strength should be about one-tenth the ultimate values given in the beginning of this chapter. Further, seeing that a compressive stress is accompanied (except when there is a lateral support) by a shearing stress of a maximum intensity equal to half the compressive stress, we may obtain safe wall and pier loads by dividing the ultimate shearing stresses by 5 (*i.e.*, dividing by 10 and multiplying by 2, to convert from shearing to compression).

	Tons per sq. ft.		Tons per sq. ft.
Granite . .	10	Sandstone .	6
Basalt . .	8	Do. (soft)	2
Slate . .	8	Limestone .	8
Marble . .	10	Do. (soft)	7

The strength of work will depend on the manner of construction and the mortar, as already explained. Thus, adverting to the rubble and granite comparison, we have 1 ton per sq. ft. as the minimum, and 10 tons per sq. ft. as the maximum working load on masonry in cement mortar.

Probably lime mortar rubble should not bear more

than $\frac{1}{2}$ ton. Fine masonry in lime mortar could bear nearly as much as cement mortar, since the mortar is not called upon to do much.

Attention should also here be called to the fact that stone is in no sense of the word approximately isotropic, as are steel and iron. It is distinctly allotropic, *i.e.*, of different elasticity in different directions. Its density is generally greatest in sedimentary rocks across the planes of sedimentation, and hence, on account of the greater resistance to compression, these planes (the "natural bed") should be laid so as to be approximately perpendicular to the direction of pressure.

A word or two may be said as to the strength of dowels, cramps, joggles, &c.

Dowels of slate are generally placed between blocks to prevent them slipping on one another. It is best to assume that the blocks are uncemented. The dowels are generally a little over 1 in. square, of slate or very hard stone. The ultimate shearing stress of such material is about $\frac{1}{2}$ ton per sq. in., so that if the total probable shearing force on the bed be x tons, there should be $20x$ rivets (factor of safety 10).

Bronze or copper cramps are generally $\frac{1}{2}$ in. or more square, turned in about 2 in. These are partly in tension and partly in bending. If the two blocks connected be pulled from each other with a force of y tons,

and the cramps are, as mentioned, $\frac{1}{2}$ in. square, the stress (tensile) in the bronze is

$$4y + 384y = 388y \text{ tons per sq. in.}$$

This stress should not exceed 2 tons per sq. in., so that only a very small pull is allowable. If, on the other hand, the metal be 1 in. square, the stress drops to $7y$, so that it is obviously better to have large cramps.

Joggles and similar interesting joints are almost wholly subject to shearing force, so that we may write

$$\left.\begin{array}{l}\text{Total shearing}\\ \quad\text{resistance of}\\ \quad\text{joggle tons}\end{array}\right\} = \left\{\begin{array}{l}\text{shearing stress}\\ \quad\text{tons per sq.}\\ \quad\text{ft.}\end{array}\right\} \times \left\{\begin{array}{l}\text{area of}\\ \quad\text{sections}\\ \quad\text{(sq. ft.)}\end{array}\right.$$

This rule is of importance in the case of lighthouse work, where very ample provision has to be made against shearing forces arising from the dynamic action of waves.

Numerous cases of bending arise in connection with arches, domes, vaults, retaining walls, chimneys, and brackets. With the exception of the last there is always, in addition to the bending, a direct loading; so that the material is subject at one place to a great compressive stress due to both bending and loading, and in another place to little compressive stress (or

sometimes tension) due to neutralisation of the loading by the bending stresses.

Since masonry joints are uncertain, and the tensile strength of stone low, it is generally as well to avoid having tensile stresses, although in some cases they cannot be avoided. In modern practice it is usual, wherever there may be tension, to build in steel beams to take it. This device may be regarded as the origin of reinforced concrete.

It should be noticed that the ability to resist stress depends in some cases on the age of the stone and its weathering capabilities. Thus some of the softer stones decay, and are disintegrated by frost, so that their compressive strength is greatly reduced and their tensile strength practically destroyed.

The cases of stress in masonry may be conveniently grouped as follows:

(1) **Walls**—subject to simple compression and oblique shearing.

(2) **Columns**—subject to simple compression, bending, and shearing.

(3) **Brackets**—subject to simple bending.

(4) **Arches**—subject to bending, compression, and shearing.

(5) **Arched structures,** including more complex stresses.

(6) **Retaining walls**—subject to compression, bending, and shearing.

The stresses may again be grouped in the following manner:

(1) *Simple compression and oblique shearing.*—Short columns and walls.

(2) *Simple bending.*—Brackets, footings.

(3) *Bending, compression, and shearing.*—Retaining walls, arches, and chimneys.

CHAPTER II

WALLS

Walls may be classified according as the loading is wholly or only partially vertical. The latter, in which the loading is partly lateral, are described as "retaining walls," and are dealt with later.

It is usual to assume that all loads (including the weight of the wall itself) are uniformly distributed over its base. It may be doubted whether this is always true, particularly when regard is had to the cases of unequal settlement on uniform soil which occasionally occurs.

The assumptions underlying this belief really apply to a wall of uniform material, uniformly bonded with mathematically plane joints on a bed of uniformly resilient soil and of infinite length.

In an actual wall of slightly irregular material, unequal bond, and roughly finished joints, there will necessarily be an inequality in the distribution of pressure.

Irregularity of bond is the most potent cause of such inequality. Thus if a load W be transmitted in any manner to one block and this be supported by two others, presupposing the joints are uniform, the load will be transmitted exactly in the proportion in which

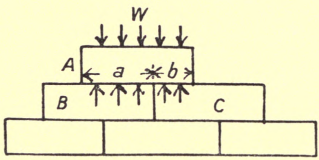

FIG. 2.

the lengths of bed joint happen to be. Thus in the sketch, the pressure on B due to A will be $\frac{a\,W}{a+b}$, where a and b are the lengths of bed on blocks B and C.

It will perhaps be objected that

(a) The adjacent blocks to A on the same course will bear similarly, so that the pressure is equalised

(b) That the blocks B and C being similarly unsymmetrically supported, will redistribute the pressure, and so on from block to block until the loading is uniform.

In perfect work undoubtedly both these objections hold good, but in practice (a) may fail by W being irregularly transmitted from above or a concentrated load bearing solely on the block A; and as far as (b) is

concerned, the slightest cavity, crack, or bump in any joint spoils the uniformity of the transmission. These secondary influences are, it is true, generally neglected in calculations, but they account for the high factors of safety which it is necessary to employ.

In most cases of walling it is convenient to consider just one foot run and study that by itself. Every similar foot will of course be under the same conditions of stress. Hence we may write two simple rules for stresses in masonry.

(1) $\underset{\text{(tons per sq. ft.)}}{\text{Max. compressive stress}} = \underset{\text{(tons)}}{\text{wt. per cubic ft.}} \times \underset{\text{(ft.)}}{\text{height.}}$

(2) Max. shearing stress $= \frac{1}{2}$ max. compressive stress.

We may deduce from these rules the maximum height of walls in rubble and granite. In the last chapter it was shown that the working compressive stress should not exceed 1 to 10 tons respectively of these stones. Now rubble weighs generally less than 1 cwt. per cubic ft. and granite about $1\frac{1}{2}$ cwt., so that the maximum safe height of a random rubble uncoursed wall is about 20 ft., and of a granite wall about 130 ft.

As a matter of fact walls very rarely fail by simple shearing or compression, since the above-mentioned limits are passed only exceptionally. The most usual causes of failure are :

(1) Foundations of irregular resistance longitudinally, leading to concentrated pressures and shearing.

(2) Foundations of irregular resistance transversely, leading to overturning of the wall and collapse by bending or actual collapse.

(3) Unequal arrangement of concentrated loads on walls, causing similarly unequal reactions at the foundations and unequal settlement.

Under the first heading we have the case which not infrequently happens where part of the foundation sinks leaving a length of walling say x ft. long unsupported. At each end of this length there are reactions supporting the intermediate load (say wht when w is weight per cubic ft., t is thickness, and h the height), so that there is a shearing force (vertically) of magnitude about $wht \div 2$. Concurrently there is a horizontal force of the same magnitude, and hence on an oblique plane of 45° inclination to the horizon there is a shearing stress of $w/2$ lbs. per square ft. Since w may be 180 lbs. or so in the heaviest stones and even the joints have a shearing strength of some 75 lbs. per sq. in. at the maximum, it does not seem probable that in decent work any failure could happen this way unless h is so small that bending stresses are appreciable. Hence we may conclude that concentrated loads (due to beams, roof timbers, &c.) are probably responsible for such apparent failures.

The second case is only of comparatively common occurrence, particularly in low walls and piers not carefully founded. Monumental masonry generally fails in this respect. If the supporting material is of unequal strength the subsidence will be greater at one point than another. If on the whole there is more subsidence on one side of the centre than the other we shall find that the wall will be slightly canted over. If the centre of gravity of the wall is thus displaced horizontally through a distance δ, there is a moment on the wall tending to accentuate the pressure on the same side of the centre, producing a further turning motion, which will, unless special precautions are taken, steadily increase with time. So soon as δ equals half the base thickness the other side of the wall is put in tension and, unless well cemented, the work will fall.

The third case has more especial reference to the building as a whole. Thus if on a wall the load per sq. ft. of base greatly exceeds that on a similar wall on the other side of the building, the structure as a whole tends to turn towards the side of greatest load. Again, an unsymmetric arch carrying great loads will cause unequal reactions on the abutments, and unless these are built in inverse proportion as to base area there will be unequal settlement and consequent canting of the arch.

Having thus considered the general stress of plain walling we may proceed to consider two other important details—foundations and footings.

Foundations of the usual set-off type are liable to fail from shearing or bending. As far as the first is

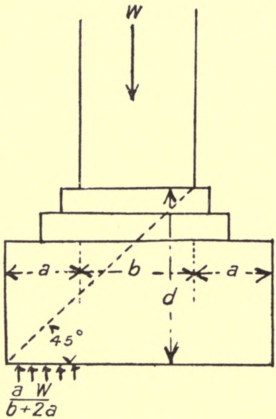

FIG. 3.

concerned it is only necessary to observe the precaution of making the depth of the under side of the foundation such that the intersection of the same with the vertical edge coincides with a plane inclined 45° to the horizon and passing through the base of the wall above the foundation (as sketch). For a given width of foundation it is obvious that the oblique resistance to shearing is a maximum and cannot be increased save by widening.

As far as bending is concerned we may take it that the distributed upward pressure on the projecting part of a foundation a (*see* sketch) $= \dfrac{aw}{b+2a}$ and the bending moment is $\dfrac{Wa^2}{2(b+2a)}$.

Hence we may write

$$\frac{Wa^2}{2(b+2a)} = f\frac{d^2}{6} \qquad (1)$$

where f is the maximum tensile stress and d the depth of the foundation immediately below the wall.

Hence

$$f = \frac{3Wa^2}{(b+2a)\,d^2} \qquad (2)$$

The working value of f allowable varies from $\frac{1}{2}$ to 3 tons per sq. ft., according to the stone employed.

Passing to the considerations of wall-ends and quoins it is important to notice that in the higher parts of the wall the oblique shearing resistance is small. Thus at the top of a high wall if there be a concentrated load of W tons at a distance x feet from the end ot the wall the shearing stress is $\dfrac{W}{2dx}$, where d is the thickness, so that this stress varies inversely as x. On this account it is necessary to build the quoins of weak walls in much stronger masonry. This is

particularly the case when random rubble or flint work is employed, the shearing strength being very small indeed. The use of angle buttresses is attributable to the same reason.

Walls are further subject to oblique forces from the arches over openings. As will be seen later, there is from each end of an arch a thrust upwards and downwards, the exact direction depending on the manner in which the arch is loaded and the form of the arch. This thrust is gradually distributed through the masonry so that eventually the whole of the lower masonry assists in the resistance. In the immediate neighbourhood of the springing the stresses are, however, more intense. Thus if the skewback thrust is T inclined θ from the vertical, there is a shearing force on the bed joints just below the springing equalling $T \sin \theta$, and a vertical pressure $T \cos \theta$. If we write

$$f_s dl = T \sin \theta - \mu . T \cos \theta \qquad (3)$$

where f_s is the shearing stress in lbs. per sq. ft., d is the thickness in feet, l the length of the wall in feet to the next wall-end, and μ the coefficient of friction (about ·7), we can calculate f_s, d, or l to suit.

If the length l is insufficient and cannot be conveniently modified, then dowels or other special shearing resistances should be employed. In this case it

will be as well to consider also whether the compressive stress is not too high. The following expression will of course give this :

$$f_c = \frac{T \cos \theta}{dl} \qquad (4)$$

where f_c is dead pressure in lbs. per sq. ft.

The failure of walls through insecure foundations has already been mentioned, but another analogous and common cause of failure is side pressure. This subject will be dealt with in detail later, but it should be here pointed out that certain lateral pressures generally exist in all walls. Thus every wall which has one or both sides exposed becomes at times subject to wind pressure, and walls forming part of a building are generally subject to certain lateral thrusts from the beams supported. The sloping rafters of a roof produce such a thrust unless there is a strong cross-tie. The common hammer-beam truss is not so secured, and consequently the walls tend to be thrust outwards. In a square-pitched roof with a hammer-beam truss there is a horizontal thrust at the eaves equal to half the total weight of the roofing supported by the truss. Thus if the trusses are 15 ft. apart, the span 30 ft., and the load per sq. ft. is about 50 lbs., there is a horizontal thrust from each truss, and on each side of it, equal to some 11,250 lbs. If the wall is 50 ft. high this means

a turning moment on the wall of 562,500 ft.-lbs. The trusses generally bear in such a manner as to spread the thrust along the wall so that the turning moment per ft. of length is 37,500 ft.-lbs. The nature of the stresses so produced will be understood when retaining walls have been studied. At present it is sufficient to say that either the wall must be inclined inwards or be made of much greater thickness. The use of buttresses solves the difficulty. At the opposite side of the wall to the truss a buttress of increasing thickness is constructed. A pinnacle above this assists to throw the resultant pressure downwards.

The use of a raking shore is almost identical with that of the buttress. Oblique thrusts generally arising from unequal settlement, warping of timbers and internal failures of joints, tend to overthrow the wall just as the pressure from roof trusses does. The exact value of these is necessarily quite indeterminate, but the moment cannot exceed the weight of the wall and half its thickness, since this moment would cause collapse. Hence with a raker weighing W lbs. that is l ft. long and inclined θ, and produces a stability moment = wt. $\times$ $\frac{1}{2}$ (distance of foot from wall),

$$M = \tfrac{1}{2} W l \cos\theta \qquad (5)$$

will resist a moment from the wall to this amount.

We may thus write, to find the number of rakers required :

$$\tfrac{1}{2}n\mathrm{W}l\cos\theta = \frac{w\mathrm{L}ht^2}{2} \qquad (6)$$

where n is the number of rakers, w the weight per cubic foot of the wall, L the length of the wall, h its height, and t its thickness.

$$n = \frac{w\mathrm{L}ht^2}{\mathrm{W}l\cos\theta} \qquad (7)$$

Thus, if the w per cubic foot of measuring is 80 lbs., the length of the wall 30 ft., the height 40 ft., and the thickness 1 ft., the rakers weighing 1800 lbs. each, and the feet being 8 ft. from the wall, 9 such rakers would be required to prevent collapse if the wall was in the last extremity. Three groups of 3 rakers would doubtless be employed in this case, discretion being used in reducing the number.

In underpinning a wall similar simple calculations may be made. Thus, if a wall of the above dimensions and weight has to be supported we write

$$m(f_c\, a) = w\mathrm{L}ht \qquad (8)$$

where m is the number of dead shores required, f_c the safe pressure in lbs. per sq. in., and a the sectional area of the dead shore in sq. ins.

The extreme stress in the fibres of the needles is found by the equation

$$f_t \frac{bd^2}{6} = \frac{2w\mathrm{L}ht}{m} .c \tag{9}$$

where b is the breadth and d the depth of the needles (all in feet), and c the distance between the centres of the dead shores. Thus, using the figures above and $9'' \times 9''$ dead shores subject to 280 lbs. per sq. in. stress (compression), we have $m = 5$. An odd number being impossible six must be employed. If they are placed at 4 ft. centre and the same size needles as shores be used, the stress works out at some 13,000 lbs. per sq. in. This is, of course, not permissible, so that at least twice as many shores and needles are required. Deepening the needles or shortening the distance between the dead shores would reduce the stress, but in this case not sufficiently.

The use of iron ties in retaining masonry walls which are insecure is also worth consideration, a similar principle to that used for dead shoring being employed. If the iron tie be x ft. above the base of the wall, and its sectional area a ins., stress f lbs. per sq. in., then we may write

$$fax = \frac{w\mathrm{L}ht^2}{2} \tag{10}$$

as in formula (6).

Thus, again employing the same figures and assuming 1 rod 2″ square at a height of 30 ft., we have $f=4000$ lbs. per sq. in.

Stresses are sometimes produced by the scaffolding employed for erection and repairs. When independent standards are used these stresses are but trifling. If, on the other hand, the scaffold is bracketed out from the work and carries heavy loads (such as unset blocks, winches, or the like), then a certain moment is exerted on the wall. If the total weight is W lbs. and the mean distance from the wall is y ft., there is, of course, a moment Wy ft. which tends to overturn the wall, the stresses being computable as for retaining walls. Similarly, any work, permanent or otherwise, built out from the wall produces a bending effect upon the wall. The overhanging wall at the quoin supported by two columns similarly produces a bending effect in itself and the adjacent masonry. These effects will be best considered in the light of the subsequent chapters.

CHAPTER III

COLUMNS AND PIERS

The conditions on which a short column or pier is placed differ but little from those appertaining to walls, but so soon as the length becomes considerable certain other problems arise which greatly increase the chance of fracture. Just as in the case of steel or wooden columns, there is a certain amount of bending which has to be kept below the "crippling" limit; but, unfortunately, the conditions are not nearly so simple or well known. We are, in the absence of better information, bound to assume that some such law as Euler, Rankine, and Gordon's still applies, and endeavour to produce a formula of the form

$$\text{Safe Load on Long Column} = \frac{\textit{Safe Load on Short Column}}{1 + \left\{\text{Constant } \kappa \left(\frac{l}{d}\right)^2\right\}}$$

where l is length and d the diameter. The safe load on short column may, of course, be deduced by doubling the safe shearing stress and multiplying by the plan area.

In the case of steel it will be remembered that the constant ("a") is $\frac{1}{2500}$ for rectangular columns, and for circular columns about $\frac{1}{2000}$. Since also in Euler's formulæ we have as the second term in the denominator

$$\frac{f_c s l^2}{4\pi^2 \mathrm{EI}}$$

(where $f_c s$ is the safe load on the short column as above, I the moment of inertia of section, and E the modulus of elasticity), we may perhaps safely say that, other things equal, this constant varies inversely as the modulus E. Now E for stone is 12 to 3 times smaller than it is for steel, so that we may write the constant say 10 times greater than it is for steel.

Hence we arrive at the two following rules:

$$\text{Safe Load on Long Column} = \frac{f_c s}{1 + a\left(\frac{l}{d}\right)^2} \qquad (1)$$

where f_c = twice the shearing stress allowable (tons),

s = area of plan (square feet),

$\left.\begin{array}{l} l = \text{length} \\ d = \text{least diameter} \end{array}\right\}$ (both in inches or feet),

$a = \frac{1}{250}$ for rectangular plans, or $\frac{1}{200}$ for circular plans.

In the case of Doric (Roman), Tuscan, Ionic, and Corinthian orders, the ratio $l \div d$ does not exceed 10, so

that we have as a general rule for circular columns

$$\mathrm{W} = \tfrac{2}{3} f_c s \qquad (2)$$

W being the total load on the column.

In slender Gothic (Decorated or Perpendicular types) columns the ratio l/d may rise to 20, or even more. For a circular shaft having this ratio we write

$$\mathrm{W} = \tfrac{1}{3} f_c s \qquad (2a)$$

In very long slender shafts the ratio may rise to 30 or 40, the latter ratio giving

$$\mathrm{W} = \tfrac{1}{9} f_c s \qquad (2b)$$

This last rule may probably be regarded as the limit.

It must be remembered in this connection that the rule presupposes the superincumbent load to bear exactly centrally over the shaft. If, owing to unavoidable causes, the centre of pressure on the head of the shaft is, say, δ distant from the centre of the shaft, a further allowance must be made for the bending moment Wδ so produced.

In this case we shall have a stress

$$' = \frac{\mathrm{W}\delta y}{\mathrm{I}} \qquad (3)$$

in addition to the above. Here I is the moment of inertia of the section ($\frac{\pi d^4}{64}$ for a circular section and $\frac{bd^3}{12}$

for a rectangular one), and y is the least distance from the centre to the face $(=\frac{d}{2})$.

Thus there is in a rectangular shaft $b \times d$ (the latter is the lesser dimension) a stress (compression on one side and tension on the other)

$$f = \frac{6W\delta}{bd^2} \qquad (3a)$$

and on a circular shaft

$$f = \frac{32W\delta}{\pi d^3} \qquad (3b)$$

If this be added and subtracted from the compression stress found by (2), (2a), or (2b), the maximum and minimum compressions will be found.*

There is a certain value in any column beyond which the displacement must not pass without tension being produced. This is found in the following manner:

Let $W = f_c s$ as in the numerator of (1), the alteration to (2), (2a), or (2b) being made by the use of a lower value for f_c. Then the stress at the edges of a rectangular section is

$$\frac{W}{s} \pm \frac{6W\delta}{bd^2} \qquad (4)$$

* The author is well aware that this method is not theoretically true for eccentrically loaded columns, but in view of the uncertainty as to column stresses in masonry it does not seem advisable to use the more accurate theoretical methods of Prof. Perry (see *Applied Mechanics*).

If on the side where dead-weight pressure is opposed by the tension due to bending there is an exact balance (*i.e.*, no real tension), then

$$\frac{W}{s} - \frac{W\delta}{bd^2} = 0.$$

But $s = bd$, so that $\delta = \frac{d}{6}$ (5)

Hence the eccentricity of the loading must not exceed $\frac{1}{6}$ the least dimension of a rectangular column if tension is to be avoided.

In the case of a circular column we have

$$\frac{4W}{\pi d^2} - \frac{32W\delta}{\pi d^3} = 0,$$

whence δ must not exceed $\frac{d}{8}$ (6)

Another interesting question in connection with a column is the maximum height. We have already seen (chap. ii.) that there is a limit to the height to which a wall may be built. In the case of a column there are two causes acting, one to increase the height and the other to diminish it. The first is the taper and the second the flexural stresses which may occur in the lower portion. As far as the taper is concerned this is so much neutralised by the entasis and the projection of the capital or superstructure that it seems hardly worth considering. As far as the flexural stresses are

concerned, the following fairly simple process should serve:

From (1) we have

$$wls = \frac{f_c s}{1 + a\left(\frac{l}{d}\right)^2} \qquad (7)$$

where w is the weight per foot cube.

Hence

$$\frac{wal^3}{d^2} + wl = f.$$

This involves the solution of a cubic equation, and this may of course be performed by Cardan's method, but as a matter of fact we generally are fixed as to the ratio $\frac{l}{d}$ by æsthetic reasons, so that it will be preferable to write

$$wl\left\{1 + a\left(\frac{l}{d}\right)^2\right\} = f_c$$

and

$$l = \frac{f_c}{w\left\{1 + a\left(\frac{l}{d}\right)^2\right\}} \qquad (8)$$

Thus if the column is to be circular and $l/d = 10$,

$$l = \frac{f_c}{w \times 1\frac{1}{2}}.$$

Thus in the case of granite, where f_c is, say, 24,000 lbs. per sq. ft. (a little over 10 tons) and w is 180 lbs., we

have $l = 100$ ft. It should be noticed that the height is not very great, and also that no wind pressure is allowed for.

It is now necessary to allow for wind pressure. This of course produces a moment, being the total wind pressure × height of the centre of pressure,* acting at the base of the shaft. This moment, say Ph, is identical in character with that produced by eccentricity, if loaded, so far as the base stresses are concerned; so that we write for a solid rectangular shaft, taking the wind at 50 lbs. per sq. ft.,

$$f_{max.} = \frac{wlbd}{bd}\left\{1 + a\left(\frac{l}{d}\right)^2\right\} + \frac{6\mathrm{P}h}{bd^2}$$

$$= \frac{wlbd^2\left\{1 + a\left(\frac{l}{d}\right)^2\right\} \pm 300\ dlh}{bd^2} \qquad (9)$$

In the case of a tapering shaft such as a chimney the weight may be computed in the following manner:

If the taper is 1 in n, *i.e.*, in n ft. of height there is a diminution in diameter of 1 foot, then the diminution in area is in ratio 1 to n^2. Hence we say, taking s as the base area, that the sectional area at 1 ft. high is

* As the author has shown in his book on the *Force of the Wind*, this height is above the mid-height of the shaft (about 60 ft. up on a 100-ft. shaft); this is due to higher velocities of the wind at greater heights.

$s\left(1-\frac{1}{n^2}\right)$, and for the second foot $s\left(1-\frac{1}{n^2}\right)^2$, so that the respective areas form a geometrical series in which the common ratio is $\left(1-\frac{1}{n^2}\right)$. The sum of these to a height h is found by the usual rule for geometrical progression.

$$\Sigma = \frac{s\left\{1-\left(1-\frac{1}{n^2}\right)^l\right\}}{1-\left(1-\frac{1}{n^2}\right)} \qquad (10)$$

This will be the volume of the shaft material, and if multiplied by the weight per cubic foot, will give the total weight.

Thus for a shaft 150 ft. high, tapering 1 in 10 and 25 ft. sq. at base, with an internal flue 10 ft. sq., we have as the total volume:

$$\Sigma = \frac{625\left\{1-\left(1-\frac{1}{100}\right)^{150}\right\}}{1-\left(1-\frac{1}{100}\right)} = -\frac{625\times\cdot 78}{\cdot 01} = 48750 \text{ c. ft.}$$

This multiplied by the weight per cubic foot (say 158 lbs.) gives the weight, nearly 3300 tons.

To apply the foregoing rules as to column stress and weight to this case we must compute the ratio l/d and also modify our bending rule to suit the form of section (no longer solid).

The shaft is 25 ft. diameter at the base, so that the ratio $l/d = 6$ (Note: d in this case may be taken as the bottom, since the weight increases with the height from the top of the shaft) and $a = \frac{1}{150}$ nearly,* so that $1 + a\left(\frac{l}{d}\right)^2$ equals say $\frac{5}{4}$, and the possible compressive stress is

$$\frac{5}{4} \cdot \frac{3300}{625} = 6\cdot\dot{6} \text{ tons per sq. ft.}$$

The wind pressure may be taken at 50 lbs. per sq. ft. on the area of one side [= mean width (20 ft.) × height], 3000 sq. ft., acting at 50 ft. up

$$= 12{,}000{,}000 \text{ ft.-lbs.}$$
$$= \text{nearly } 5400 \text{ ft.-tons.}$$

In the case of a hollow square section the formula (3a) becomes

$$f = \frac{6 \times \text{moment} \times d}{d^4 - d_0^{\,4}},$$

where d_0 is the internal diameter.

This works out to

$$\frac{6 \times 5400 \times 25}{(25^2 + 10^2)(25^2 - 10^2)} = 2\cdot04 \text{ tons per sq. ft.}$$

Hence the maximum compression in this case is $6\cdot\dot{6} + 2\cdot04 = 8\cdot7$ tons per sq. ft., and the minimum is $6\cdot\dot{6} - 2\cdot04 = 4\cdot62$ tons per sq. ft. There is against this

* This follows from the method given at the beginning of the chapter.

method the objection that the taper is not generally uniform and that entasis is important. This is true, but, on the other hand, a carefully obtained mean taper will give results which do not differ greatly from those obtained by more laborious treatment.

As before, the deviation of the resultant from the centre of the base without producing tension is calculable by writing, as in (4),

$$\frac{W}{s} - \frac{6W\delta.d}{d^4 - d_0^4} = 0 \qquad (11)$$

We should notice here that $W\delta = Ph$, and that $\delta = d^2 - d_0^2$, so that this expression simplifies to

$$6\delta.d = d^2 + d_0^2,$$

so that

$$\delta = \frac{d}{6} + \frac{d_0^2}{6d}.$$

If $d \div d_0 = m$, so that $d_0 = \frac{d}{m,}$ we have

$$\delta = \frac{d}{6} + \frac{d}{6m^2} \qquad (12)$$

It should be noticed that the deviation becomes more as the flue increases, until finally, when the shell is very thin, a deviation $= \frac{d}{3}$ is permissible.

A similar method may be adopted with hollow circular shafts, making the proper substitutions for the

area and moment of resistance (as for hollow steel columns), but it should be noted that the wind pressure per square foot of projected area of side will not exceed half that allowed for a flat-faced shaft. On an octagonal shaft rather more than half should be allowed.

The nature of the stresses in columns forming part of large buildings needs a little consideration. The effect of eccentricity in loading has been already pointed out. Not less important are the bending moments transmitted by beams rigidly connected with the heads of columns. This case does not frequently happen in masonry, but in monolithic concrete and reinforced concrete structures it is usual. If one column have a beam passing continuously over it and the loading is symmetrical, no bending should occur in the column. On the other hand, if the beam stops at the column and *is tied down by cramps to it*, a moment is produced in the column. If the load on the beam is 20 per foot run for a span of l ft. the end moment is $\frac{wl^2}{12}$. This moment must be regarded as similar in effect to a moment caused by eccentric loading.

If a column or pier is not straight, or if the materials of which it is built are not uniformly elastic, we have the case of the curved rib, which rapidly develops into the arch when lateral pressures are considered.

A curved rib, whose chord is l feet long and whose versed sine is d feet, resisting a thrust T, becomes immediately subject to a bending moment at the centre$=$Td. This tends to slightly increase on account of the deflection produced, so that we should write $M=T(d \times \delta)$. This presupposes that the thrust acts centrally through the ends. If the rib is fixed at the ends and is comparatively short, the deflection δ will be negligible.* The stresses may then be computed just as before, employing Td instead of the eccentricity or wind moment. When there is lateral pressure the thrusts may have innumerable positions relative to the centre line, as will be explained in the chapter on arches.

It is, however, sufficient to remark here that a curved rib differs only from an arch in that the arch is so

* δ can be approximately computed by the curvature law $\frac{d^2y}{dx^2}=-\frac{M}{EI}$, if we assume the curvature is parabolic so that the bending moment at x from the centre$=T(d-\kappa x^2)$.

[Note that $d=\frac{\kappa l^2}{4}=0$, so that $\kappa=\frac{4d}{l^2}$.]

$$\frac{dy^2}{dx^2}=\frac{T(d-\kappa x^2)}{EI}=\frac{Td}{EI}-\frac{T\kappa x^2}{EI};\ \frac{dy}{dx}=\int\frac{d^2y}{dx^2}=\frac{Tdx}{EI}-\frac{T\kappa x^3}{3EI}+C_1,$$

$$C_1=0;\ y=\int\frac{dy}{dx}=\frac{Tdx^2}{2EI}-\frac{T\kappa x^4}{12EI}+C_2;\quad C_2=0;$$

Hence $\delta=\frac{Tl^2d}{EI}\left(\frac{1}{2}-\frac{1}{3}\right)=\frac{Tdl^2}{6EI}$. The effect of δ in increasing itself here may be disregarded.—H. C.

arranged that the thrust shall pass axially through it, whereas in a curved rib the thrust is generally along the chord of the axis.

The method of bedding the ends of a curved rib is rather important. The deflection will be considerably greater if the ends are fitted with a hinged or circular joint than if there is a square butt joint. Owing to friction the difference is not so great as might be expected, but it certainly exists.

The question of buttresses and retaining walls, which are analogous to the column, is discussed in a later chapter.

CHAPTER IV

BRACKETS AND CANTILEVERS

ALTHOUGH it is unusual to construct this type of structure in masonry, cases do occur, and the study of them will lead usefully to the more complex problems

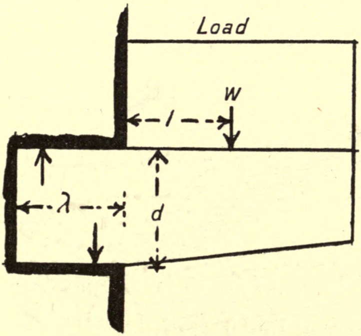

FIG. 4.

involved in arches and retaining walls. In these cases there will occur that which is generally prohibited, viz., tensile stress in the stone (Fig. 4).

The general problem is the same as that of the rectangular beam. A bending moment, Wl, is pro-

duced, resisted by a moment of resistance, $fbd^2 \div 6$, so that we have, as the maximum tensile or compressive stress,

$$f = \frac{6Wl}{bd^2} \qquad (1)$$

From the values for the ultimate tensile stress given in the first chapter, it will be seen that the safe tensile stress varies from $\frac{1}{2}$ to 8 tons per sq. ft. A safe value for tough stones will be 2 tons. Since the safe compressive stress (shearing effect also being considered) averages much more than this (say 6 tons), we may with economy make the section of the cantilever larger at the top than below, the area above the neutral axis being about three times that below it.

There is also a shearing effect at each point between the load and the support. In the case illustrated, the shearing effect is equal to the load at the support, diminishing to zero at the end of the beam. Since the shearing resistance of stone is comparatively small, this should receive as much consideration as the bending. The stress produced in the beam by this shearing force is most intense at the centre of the section (the neutral axis). In the case of a rectangular beam it there reaches the magnitude of $\frac{3}{2}$ times the mean shearing stress, which latter is found by dividing the shearing force by the area of the section.

In the case of a triangular section, the apex of the triangle being downwards, the top breadth b and the depth d, the extreme stress in the top side may be found by dividing the moment by $\frac{bd^2}{12}$, and the stress on the bottom may be found by dividing the bending moment by $\frac{bd^2}{24}$. The latter will, of course, be twice as great as the former. If the beam acts as a bracket, the first will be in tension and the second compression:

$$f_t = \frac{12Wl}{bd^2} \qquad (2)$$

$$f = \frac{24Wl}{bd^2} \qquad (2a)$$

Actually triangular sections are rare, but many sculptural brackets are approximately triangular in section; so that if the dimensions (less all work in relief) be taken as for a triangular section, the stresses may be computed.

As an example, let us suppose a stone bracket supporting a corner turret weighing 5 tons has a projection of 10 ft., the weight acting at the centre of the span. Then if the bracket be 4 ft. wide and 5 ft. deep (triangular), the maximum tensile stress is $= \frac{5 \times 5 \times 12}{4 \times 25} = 3$ tons per sq. ft., which is rather

high. Joints without numerous cramps will, of course, be impossible, and the bonding in is very important.

This latter point is one very much neglected, leading occasionally to crushing of the wall in the neighbourhood of the bracket, and sometimes to failure of the wall by bending. The bending moment is, of course, transmitted to the wall with undiminished magnitude. In fact, it is increased by the weight of the bracket itself. Hence the wall must be guarded against failure by bending, as is explained later in dealing with retaining walls. Further, it must be recognised that the inner end of the upper side of the bracket tends to rise, and the outer end of the under side to fall, so that the upper end of the bedding-in is compressed upwards, and the under face downwards. It is usual to assume that these pressures are simply proportional to the length of the inset, the upper one increasing from the face of the wall to a maximum at the inner end of the bracket, and the lower pressure increasing from zero at the inner end of the beam to a maximum at the face of the wall. The edge of the under block, or template, is usually chamfered to prevent spalling at this edge. If, as is the case usually, the built-in part is rectangular in section—b wide, d deep, and λ long—then we have the pressure on each side spread over an area, $b\lambda$, the mean value being found at $\frac{1}{2}\lambda$, the value at one end of

λ being 0, and at the other twice the mean value. The centre of pressure on each face is then $\frac{2}{3}\lambda$ from one end of λ, and the distance between the centres of pressure on the two faces is $\frac{\lambda}{3}$, so that if the total pressure on each face is p, then $\frac{p\lambda}{3}$ = the bending moment = Wl.

But p = the mean pressure (tons per sq. ft.) $\times\, b \times \lambda$, which again = $\frac{1}{2}$ max. pressure $\times\, b \times \lambda$, so that we may write

$$\frac{p_{max.} b\lambda}{2} \cdot \frac{\lambda}{3} = Wl \qquad (3)$$

and hence

$$\lambda_2 = \frac{6Wl}{p_{max.} b} \qquad (3a)$$

or

$$p_{max.} = \frac{6Wl}{b\lambda^2} \qquad (3b)$$

From these rules we may proceed to find the length of the inset, or knowing this, we may find the maximum compression.

Using the figures given above and assuming a rectangular inset 4 ft. wide by 5 ft. deep and 5 ft. long, we get

$$p_{max.} = \frac{6 \times 5 \times 5}{4 \times 25} = 1\tfrac{1}{2} \text{ tons per square foot.}$$

Several other questions in connection with masonry

brackets are of importance. Thus a large console or truss decorated like a Corinthian modillion everywhere approximately rectangular in vertical section needs no further strengthening if its load is distributed and the section is sufficient at the support, for the line of necessary depth will pass well within the mass of stone above the scroll. On the other hand, if there is an end load, care must be taken that the shallowest part (just behind the lesser scroll) is sufficiently deep to allow an area for shearing resistance.

This latter consideration will, as has been mentioned, generally be the most important in endeavouring to economise masonry. The forms of least material commonly employed for cast iron, steel, and timber do not exactly apply to masonry for this reason.

There can of course be no joint in a bracket or cantilever unless metal cramps, dowels, or ties of some kind are used at the places where tension occurs. Thus if a bracket has for some reason to be made up with small blocks, every top joint must be cramped, and every vertical joint joggled. The effect of substituting ties or cramps for stone in tension is to lower the neutral axis to about two-thirds the depth, so that less stone is available to resist compression.

Thus the moment of resistance becomes:

$$f_t\, a\left(\frac{2d}{3}\right)+f\frac{b\left(\frac{d}{3}\right)^2}{3}=\frac{2f_t\, ad}{3}+\frac{f_c\, bd^2}{27} \qquad (4)$$

where f_t is the tensile stress in the metal, a the sectional area, d the depth of the stone, b the breadth (rectangular section), and f_c the compression in the stone.

It will thus be seen that a jointed cantilever is far less efficient than a solid one, so that generally it will need to be larger and the economy will be lost.

The stresses in this case are closely analogous to those which occur in reinforced concrete, to be dealt with later, save that in the latter no part of the concrete is regarded as resisting (effectively) tension, whereas here the stone between the joints does resist tension, the magnitude of the latter being at the upper edge of any vertical section nearly twice that of the compression at the lower edge of the same section.

Another important type of bracket is that which is tee-shaped in section, the table of the tee being uppermost. The best proportion for this is one which brings the centre of gravity of the section level with the under side of the table. Given the dimensions, we know that the centre of gravity will lie on the line of symmetry at a distance from the upper surface.

$$x=\frac{d_1+d}{2}\cdot\frac{b_1d_1}{b_1d_1+bd}+\frac{d}{2} \qquad (5)$$

If the condition mentioned is to be fulfilled, then $n=d$, so that

$$\frac{d_1+d}{2}\cdot\frac{b_1d_1}{b_1d_1+bd}=\frac{d}{2}.$$

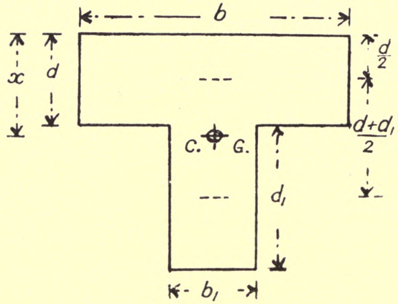

FIG. 5.

Multiplying across, we have

$$(d_1+d)(b_1d_1)=d(b_1d_1+bd),$$

so that the condition that the centre of gravity shall be in the required place is that

$$b_1d_1^2=bd^2,$$

or that

$$d=d_1\sqrt{\frac{b_1}{b}} \tag{6}$$

The moment of inertia of the section about an axis through the centre of gravity in this position is

$$(bd+b_1d_1)\div 3$$

so that the maximum tension

$$f_t = \frac{3Md}{bd + b_1d_1} \tag{7}$$

and the maximum compression

$$f_c = \frac{3Md_1}{bd + b_1d_1} \tag{7a}$$

It should be noticed that the maximum shearing stress occurs at the level of the neutral axis, so that the upper

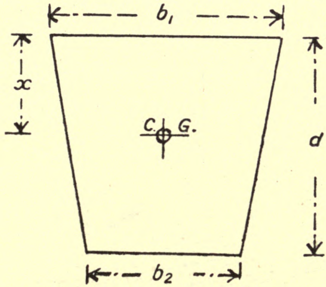

FIG. 6.

block must be well keyed to the lower. Several ornamental forms of bracket closely approximate to this section.

Yet another type of section not infrequently met with is a symmetrical trapezoid, b_1 wide at top and b_2 at the bottom.

The neutral axis is situated at a distance x from the upper surface

$$= \frac{3b_1 + b_2}{3(2b_1 + b_2)} \cdot d \tag{8}$$

The moment of inertia is

$$I = \frac{6b_1^2 + 6b_1b_2 + b_2^2}{36(2b_1 + b_2)} \cdot d^3 \qquad (9)$$

so that the maximum tensile stress (top) is

$$f_t = \frac{Mx}{I} = \frac{12(3b_1 + b_2)M}{(6b_1^2 + 6b_1b_2 + b_2^2)d^2} \qquad (10)$$

where M is the bending moment.

As an example of this case, let us assume that a cantilever of the form given is 4 ft. wide at the top and 2 ft. at the bottom (inside measurements, excluding all carving) and 5 ft. deep, 10 ft. projection, carrying a distributed load of total amount 10 tons.

$$f_t = \frac{12(3\times4+2)(10\times5)}{(6\times16+6\times4\times2+4)25} = \frac{8400}{3700} = 2{\cdot}3 \text{ nearly (tons per sq. ft.).}$$

It is interesting to notice that in all cases the work may be jointed without danger below the neutral axis, and for economy of material an arched form may be used for the under side, as already mentioned.

When terra-cotta or other hollow material is employed instead of masonry, it will be preferable to disregard the strength of the cement filling, and compute as if perfectly empty. This leads then to another type of beam, whose section is a hollow rectangle.

The moment of inertia of a symmetrical hollow rectangle is

$$\frac{bd^3 - b_1d_1^3}{12} \qquad (11)$$

where b_1 d_1 are the inside dimensions.

The stresses work out to

$$f_t = f_c = \frac{6dM}{bd^3 - b_1d_1^3} \qquad (12)$$

Terra-cotta brackets will necessarily be of small projection, and the following case will illustrate the need of careful proportioning in practice.

Material 2 in. thick, projection 2 ft., load (spread) 2 tons, outside breadth 1 ft., depth $1\frac{1}{2}$ ft.

$$f_t = \frac{6 \times 1\frac{1}{2} \times 2 \times 1}{(1 \times 1\frac{1}{2}^3) - (\frac{2}{3} \times 1\frac{1}{6}^3)} = \frac{18}{2{\cdot}316} = 7{\cdot}7 \text{ tons per sq. ft. (tension),}$$

which is a dangerous stress for this material.*

By reversing the process already employed for finding the stresses, we can determine how much projection can be given to a cantilever.

Thus a rectangular bracket with a projection of $2l$ ft., bonded uniformly, is subject to a stress

$$f_t = \frac{6Wl}{bd^2},$$

so that

$$l = \frac{bd^2f_t}{6W}, \text{ or the whole projection } 2l = \frac{bd^2f_t}{3W} \qquad (13)$$

* The load on this bracket should not be more than half a ton ; the stress would then be 1·94 tons per sq. ft.

Thus if the maximum tensile stress allowable is 1 ton per sq. ft., and the load 10 tons, b being 2 ft., and d up to 3 ft.,

$$2l = \frac{2 \times 9 \times 1}{3 \times 10} = \tfrac{3}{5} \text{ ft.}$$

Similarly, the maximum load on a beam of given projection, $2l$, may be computed as follows:

$$W = \frac{bd^2f_t}{6l} \qquad (14)$$

Thus, assuming a maximum stress of $\frac{1}{2}$ ton per sq. ft. and the following dimensions: 2 ft. breadth, 3 ft. depth, 6 ft. projection ($l = 3$ ft.), we get:

$$W = \frac{2 \times 9 \times \frac{1}{2}}{6 \times 3} = \tfrac{1}{2} \text{ ton.}$$

It will perhaps be useful also to refer to the effect of a gallery on the supporting brackets. If the latter be arranged at a distance s ft. from centre to centre, the load on each is $2wls$, where $2l$ is the projection and w is the load per sq. ft. Thus, if the flooring and the load of a gallery amount to about 2 cwt. (say 250 lbs.) per sq. ft. and the cantilevers are 5 ft. apart, the total projection being 6 ft., we have a total load

$$W = 2 \times 250 \times 3 \times 5 = 7500 \text{ lbs.}$$

The moment of this is $7500 \times 3 = 22{,}500$ lbs.-ft. I

the brackets are 4 ft. deep, and 3 ft. wide at the wall, we have

$$f\frac{bd^2}{6} = 22{,}500.$$

$$f = \frac{6 \times 22{,}500}{3 \times 16} = 2812 \text{ lbs. per sq. ft. (little over 1 ton).}$$

Beams.—In concluding this chapter it will be useful to point out that the few cases in which masonry slabs serve the purpose of beams may be treated in a manner strictly analogous to that here employed for brackets.

The moments may be computed as equated to the stress moment of the section just as is here done. It is scarcely necessary in a work of this kind to remind the reader that the bending moment in the centre of a supported beam with a central bond W is WL/4, where L is the whole span, and with a distributed load WL/8, or a concentrated load not central $\frac{Wab}{L}$, where a and b are the distances from either end of the beam.

If the ends are fixed down, the bending moments, as supported, are reduced by the mean value so that WL/4 becomes WL/8, and WL/8 becomes WL/24 at the centre and WL/12 at the ends, and so on. Finally,

it should always be remembered that the stress due to bending is found by the rule

$$\text{Stress (tons per sq. ft.)} = \frac{\text{Bending moment (ft.-tons)} \times \text{Distance of the point considered from the neutral axis (ft.)}}{\text{Moment of inertia of section (ft. units)}}$$

Tension, of course, must be principally taken into account.

CHAPTER V

SIMPLE ARCHES

The theory of the arch has always proved one of the most difficult problems of applied mechanics, for the reason that it involves the consideration of internal stresses without any definite *point d'appui*.

It is usual to assume, in the first place, that the blocks of which an arch is built are not cemented one to another, but simply sustaining by mutual action and friction. How great a margin this leaves will easily be realised by those who have tried to put together an arch without mortar. It is exceedingly probable that the margin thus allowed is over ample, since many arches theoretically unsafe have proved stable. On the other hand, when an arch is large cement is far less important than initial stability.

To illustrate the difficulty which one experiences in studying the question we will first consider the case of a simple wedge, dropped into a tapered slot and weighted (Fig. 7). By the principle of the balance of

force we know that reactions are experienced from either side which neutralise one another and the weight. If both surfaces are perfectly smooth and inclined at exactly the same angle away from the vertical the reactions will be equal. If the loading be slightly

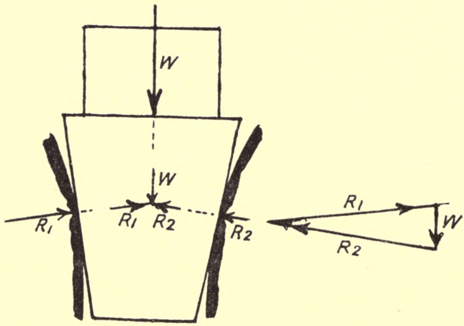

FIG. 7.

eccentric they will be unequal. If either of the surfaces be slightly irregular or inclined at a different angle from the vertical the reactions will be unequal. If one side surface be at all softer than the other the reactions will be unequal, and in all cases under practical conditions it is impossible to say where the resultant on each side acts. Assumptions obviously must be made, and if possible allowed for afterwards. These will shortly be given, our first step being to study the general conditions of things when a number of wedges are set between one another instead of a single wedge. (Fig. 8.)

Thus, let us suppose a small arch consists of three voussoirs, and that the upper surfaces are horizontal and loaded with weights W_1, W_2, and W_3. The

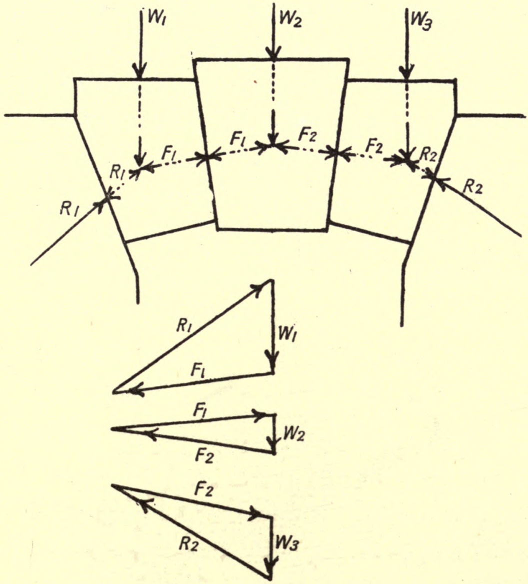

Fig. 8.

central block (key-stone) is in the same condition as the single wedge before considered. Two forces situated somewhere across the faces are balancing the bond W_2. These lateral thrusts must of course be transmitted to the next block, so that we have F_1 and a second reaction from the other face R_1; and similarly on

the other side F_2 and R_2 balance W_3. There is thus a line of thrust $R_1F_1F_2R_2$ which passes through the blocks. If the arch were a simple thin rib and this rib *exactly followed the line of thrust* it is obvious that the arrangement would be stable. Such a rib is called the "linear arch." It is obvious that a slight alteration in the loading will modify the line of thrust so that each particular set of loads necessitates a new linear arch, and the simple rib will therefore be structurally insufficient. On the other hand, if the said linear arch lies within the blocks and the forces do not exceed the crushing strength (or rather the working strength) of the blocks it will follow that the arch is stable. If the linear arch passes outside the blocks, bending and consequently tension on one side will occur, and the arch will break down.

Our great object, then, in studying an arch is to say where the line of thrust falls, since from the magnitude of its compartments the *pressure* between the blocks is thereby determined, and by the position of the line at any joint the distribution of the pressure (*i.e.*, the *bending effect*) is ascertainable.

Unfortunately, as has been pointed out in the case of a single wedge, the exact position of the line is quite indeterminate, but we may approximate to it in the following manner :

It is obvious that if two points in a symmetrical line of thrust or three points in an unsymmetrical one can be found, all other points will be determinable. Any doubts the reader may have as to this will be solved shortly.

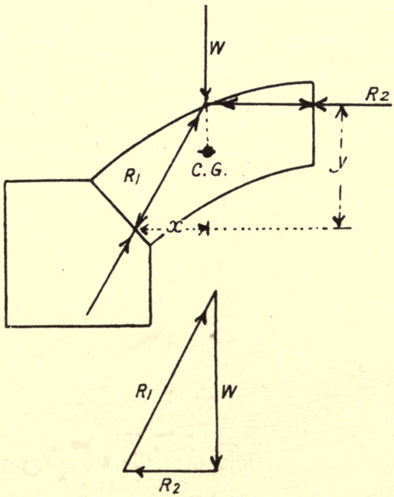

FIG. 9.

Considering first a symmetric bond, these two points will be most conveniently on the centre line and the skewback. Now there is a mechanical law known as "Moseley's principle of least resistance" which states that when there is any choice in the values of resistances to balance an entire force these resistances will be the least possible. Let us now inquire what position for

the thrusts at crown and skewback give the least values, the load being constant. Take half the rest and find the resultant bond (Fig. 9); assuming that R_1 passes through the centre of the skewback, we see that

$$R_2 y = Wx + R_2 = \frac{Wx}{y},$$

so that in order that R_2 shall be as small as practicable y must be a maximum. In other words, the higher the crown thrust the less is its actual value.

Again, assuming that R_2 passes centrally through the section, the more inclined R_1 is to the horizontal the less is its value. In other words, x should be as small as possible and y as large as possible.

This idea may be again stated in the following form: The nearer the crown thrust is to the extrados, and the nearer the skewback thrust is to the intrados, the less will each be in magnitude.

The further conclusion may be drawn that so long as the line of thrust lies within the arch face it should be as steep in mean slope as possible, so that R_2 is high and R_1 low.

It should be here noticed that R_2 is the horizonta component of R_1 and W is the vertical component. This applies to any line of thrust, R_2 being the horizontal component everywhere, so that we have the

further conclusion that the horizontal thrust in an arch is everywhere the same and is equal to the crown thrust.

It might be supposed from the above rule as to the positions of the crown and skewback thrusts that the former might safely rise to the extrados and the latter fall to the intrados. It will, however, be shown in the chapter on retaining walls that if the resultant thrust passes outside the middle third of the arch tension will be experienced on the side remote from the thrust. This is very undesirable, so that a further law is expressed as follows :

The line of thrust should not pass outside the middle third of the arch.

The maximum compression will always occur on the same side of the centre line as the line of thrust, and of course the minimum compression on the other side. If the maximum compression exceeds the crushing strength of the stone, the arch will fail by crushing. Another point to be noticed is that if the line of thrust is inclined at less than the angle of friction to any joint the arch will fail by the blocks slipping on one another.

It will be seen from the above-mentioned rules how important the line of thrust becomes, so that our whole endeavours are directed towards finding it. Hence

our next step will be to find how it may be drawn without reference to the arch itself, and then endeavour

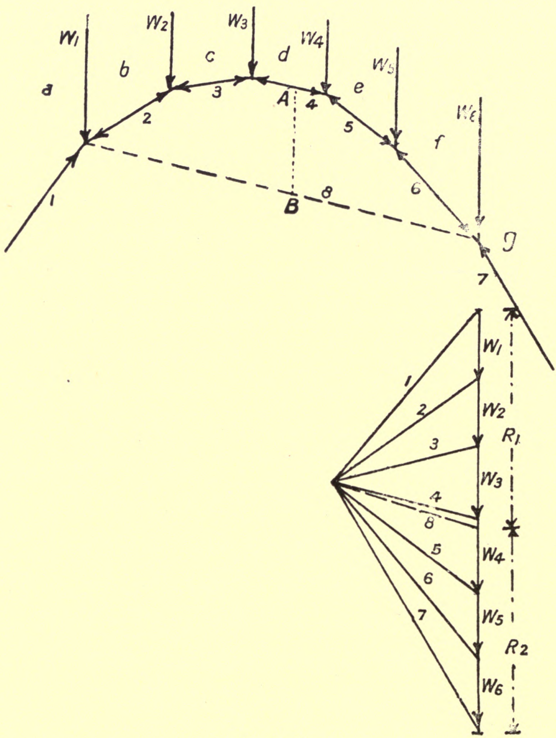

FIG. 10.

to adapt it to the arch. Subsequently we shall try to find the stresses produced.

Let us suppose the load on the arch is composed of

elements $W_1 \ldots W_n$, say six in number (Fig. 10). Draw a vector polygon for these loads, and take any point *o* to the left of the line, and join *n* rays to the end of the vectors. Then across the corresponding spaces between the loads draw parallel lines. So we have a "link polygon," 1234567, which is a line of thrust, but not necessarily that for the arch in question, since the height at the centre is purely arbitrary, depending on the distance between *o* and the vertical line of vectors. Let W_1 and W_6 be the loads on the abutments, *i.e.*, scarcely affecting the arch and joints 2 to 6 at the feet of W_1 and W_6 as shown by a line 8. Draw a line through *o* parallel to 8; then R_1 and R_2, cut off the line of vectors, are the vertical components of the skewback thrusts on either side.

Now we have to endeavour to apply this line to the actual arch. Suppose the height from the springing to the centre line at the crown is Y_1 ft., and on the scale used for the link polygon, AB (the height from the chord of the polygon to one of its sides in the centre of the span), is Y_2 ft. Then if the point *v* be brought nearer or farther from the line of vectors in the ratio $\frac{Y_1}{Y_2}$ (according as Y_1 is less or more than Y_2), and the processes of drawing repeated, we shall have a link polygon passing through the centre of the crown block

or keystone. If instead of taking Y_1 as the height of the centre line we take it to the upper edge of the middle third, the crown thrust will then pass through the point of maximum elevation allowable. If at the same time W_1 and W_6 be so chosen that they pass through the inner edge of the middle third at the springing, then the condition as to position of the skewback thrust is satisfied.

Supposing that under these circumstances the line of thrust nowhere passes outside the middle third, we assume that the arch is perfectly stable. This does not necessarily mean that it is strong enough, only that it is balanced.

Professor Fuller has devised a very ingenious method of modifying the link polygon to pass through as near as possible to these points at the crown and skewback, which will be found in Perry's *Applied Mechanics* and Rivington's *Notes on Building Construction*, Pt. IV., but if the above be carefully studied there will be no great difficulty in performing the operation.

Having proceeded so far, we must now deduce the stresses in the arch from our given polygon.

Let us suppose we have any section plane or joint in the arch AB, and the line of thrust is there inclined at θ to the section, and that the magnitude of the thrust there is P (Fig. 11). Note that $P = T \sec \psi$, where T

is the horizontal (= crown) thrust, and ψ the angle the thrust makes with the horizontal.

(Note that $\psi + \theta = w$, the angle which the section plane AB makes with the horizontal.)

Resolve P into components perpendicular and

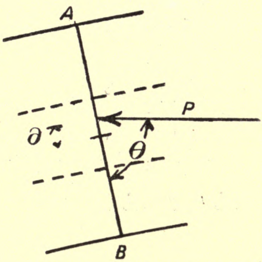

FIG. 11.

parallel to AB; P sin θ is the pressure on the joint or section plane, and P cos θ the sliding or shearing force.

Also if the force P crosses the section at a distance δ from the centre, then there is a turning moment P sin $\theta \times \delta$ on the joint.

Hence we arrive at the stresses, considering 1 ft. depth of arch,

$$\text{Maximum compression} = \frac{\text{P} \sin \theta}{\text{AB}} + \frac{6\text{P} \sin \theta . \delta}{\text{AB}^2}.$$

(*See* chap. viii.)

This may be more conveniently written

$$\text{Maximum compression (lbs. per sq. ft.)} = \frac{P\sin\theta}{AB}\left(1 + \frac{6\delta}{AB}\right) \qquad (1)$$

P must be given in lbs., and δ and AB measured in feet. The minimum compression is the same, changing the sign, but of course this is of little importance.

The maximum shearing stress is usually taken as

$$\frac{3}{2} \times \frac{P\cos\theta}{AB} \qquad (2)$$

Professor Karl Pearson has adduced reasons for doubting this, but at present the rule must stand. (*See* chap. viii.)

In some cases the arch has been formed with hinges at crown or abutments or both, so that the thrust must pass, disregarding friction, through the centre of the joint there. This, of course, avoids bending moment at these places.

As has been said, the line of thrust should not pass outside the middle third. If in a design it is found that the arrangement of loads is such that the line will not fall within the middle third there are four courses to pursue:

(1) Alter the loads.

(2) Deepen the arch.

(3) Alter the shape of the arch.

(4) Tie the extrados or intrados (as required) with clamps.

For constructional reasons it will generally be inconvenient to adopt the first line of action, and if a certain scheme of architectural treatment has been adopted, the third may be objectionable. The fourth is only used in extreme cases.

Hence deepening the arch is the most usual procedure. The steepest link polygon having been drawn, we must deepen the arch so that the new middle encloses the line of thrust. On the new arch the stresses will of course be less, since the area over which they are spread is greater. On the other hand, if the dimensions and weight of the arch are appreciable in relation to the load supported, the increased weight of the voussoirs must be considered, and a new link polygon drawn.

Three cases of failure are generally noted in regard to arches.

(*a*) The line of thrust passes outside the middle third at the haunches, causing these to sink in and the crown to rise.

(*b*) The line of thrust passes inside the middle third at the haunches, causing them to burst out and the crown to sink.

(*c*) A third case not uncommonly happens, viz.,

shearing of the arch due not to direct shearing, but compression. This is of course analogous to column stress, and may be studied in the same manner.

One consideration which is somewhat neglected is the shape of the extrados. When the voussoirs are

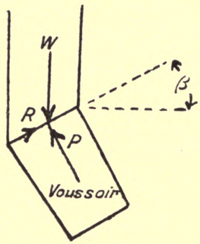

FIG. 12.

stepped, it is quite legitimate to regard the load as vertically transmitted to the voussoirs, although here the precaution should be taken of carefully finding the centre of gravity of the mass of work above and including the voussoir.

When, however, the extrados is curved (as in ordinary elliptic, semicircular, and segmental arches) it should be noticed that the mass of the spandrel above any voussoir or set of voussoirs does not bear squarely on to the same.

Thus if W (Fig. 12) be the mass of the spandrel bearing on a voussoir whose extrados has a mean

slope β, there is a tangential force $R = W \sin \beta$ and a normal force $W \cos \beta$.

When β is less than the angle of friction of course this is immaterial, the resultant reaction being necessarily vertical and equal to W, but near the springing a point must come where β exceeds the angle of friction and there is shearing force acting on the extrados. It is of course true that the adjacent parts of the spandrel tend to neutralise this, but at the same time the point should receive consideration.

Another matter of importance is the angle of the skewback. It will be found that as this angle decreases, *i.e.*, as the arch becomes a larger segment of a circle, so it is more and more difficult to fit the link polygon into the arch, particularly in the truly circular forms. It is this fact which has led to the adoption of the elliptic arch. Moreover, when the skewback has become horizontal, as in semicircular, semi-elliptic, or Gothic forms, it is rarely possible to keep the line of thrust on the intrados side of the centre.

Also from the very fact of the existence of a horizontal thrust throughout it is not possible that the skewback thrust should be vertical (*i.e.*, perpendicular to the joint), for that would imply either an infinite load in proportion to that thrust, which is absurd, or an infinitely high arch (which is nearly realised in lancet-forms).

Hence in any arch, vault, dome, or similar structure, the abutments tend to be forced apart, the outward push being the same as the crown thrust. For this reason the end arch of an arcade needs to be buttressed or strengthened in some such manner. By excessively loading the abutment (as is done in the buttresses which support "flying buttresses") it is possible to throw down the thrust, but it must always be understood that the horizontal thrust cannot be balanced except by friction and shearing resistance in the bed-joints of the abutment.

In the case of arched bridges, such as London Bridge and Waterloo Bridge over the Thames, there is a peculiar type of loading, viz., decreasing from abutment to crown, on account of the work being brought up to an approximately level surface a little above the arch. In this case it will be found that the link polygon for the loads (splitting the distributed load into a convenient number of parts) will be almost elliptic in form, so that it will fit into an elliptic arch with great ease. This fact, together with the necessity for minimising the number of abutments, has led to the use of elliptic arches in river and railway practice.

In the ordinary case of an arch supporting a wall it is a very common practice to assume that the whole of the uninterrupted mass above is supported. As a

matter of fact the bonding tends to support overhanging stones, so that the actual mass likely to fall should the arch be removed is bounded by two diagonal lines running zigzag along the vertical and horizontal joints upwards from the ends of the arch, forming a roughly triangular piece. In brick (quarter bond) structures these lines will slope in the ratio 3 to $2\frac{1}{4} = 4$ to 3, so that over a straight arch B ft. along the extrados there is a triangle $\frac{4}{3} \cdot \frac{B}{2} = \frac{2}{3}B$ ft. high, the total area of the face being $\frac{1}{3}B^2$ sq. ft.

This fact combined with the tensile strength of mortar (which is not greatly tried in this case) will account for many cases in which arches obviously unsuitable and insufficient to support a large piece of work stand for many years.

If a building which is underpinned for the purpose of putting in a shop-front be examined, it will be seen how much is due to the strength of the mortar and how little *may* be due to arches supporting work over openings.

CHAPTER VI

VAULTS AND SKEW ARCHES

(Including a Note on Simple Types of Load on Ribs)

THE principles described in the previous chapter are equally applicable to all arched structures, but some difficulty arises in dealing with the cases of vaults and domes. The different constructions may be conveniently grouped in the following manner:

(1) Barrel vaults.
(2) Gothic vaults.
(3) Skew vaults.

In the case of a straight barrel vault, each foot run may be regarded as an arch of that thickness and be studied in the same manner. At an intersection, however, certain peculiar features must be considered, since the whole of the voussoirs and superincumbent load over the crossing is transmitted to the quoins of the abutments through diagonal ribs. As it is unusual to

construct a vault in this fashion unless the spans are equal we will assume that this is the case.

Taking any section, such as AB or A_1B_1, we find a

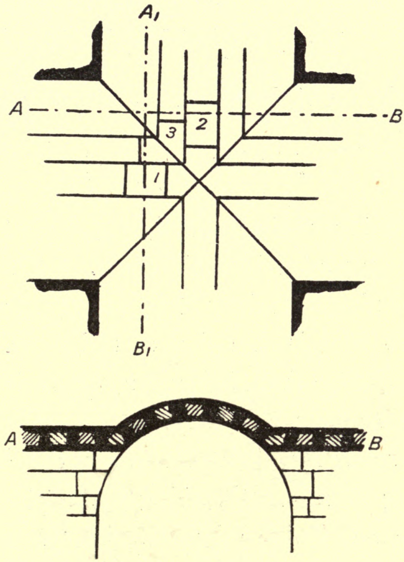

FIG. 13.

segmental arch bearing at the ends on to skewbacks formed on the voussoirs of diagonal ribs. Thus the blocks 1 and 2 in the diagram transmit their thrust to the intersection block 3. If, as is supposed, the thrust from 2 and 1 are equal and equally inclined (say T inclined θ from the vertical), then the horizontal

components will combine to produce a resultant horizontal thrust $= \sqrt{2}T \sin \theta$, and the vertical components will produce a resultant vertical pressure $= 2T \cos \theta$. The resultant of these two

$$T_{\Delta} = 2T\sqrt{\tfrac{1}{2} \sin^2 \theta + \cos^2 \theta} \qquad (1)$$

The deviation of the thrust T_{Δ} is such that

$$\phi = \tan_{-1} \left(\frac{\tan \theta}{\sqrt{2}}\right) \qquad (2)$$

where ϕ is the angle made by T_{Δ} with the vertical. From this expression it is obvious that ϕ is less than θ, so that the diagonal rib must be deeper than the barrel vaulting to contain the line of resistance. The fact that T_{Λ} is greater than T also necessitates this, by reason of the greater pressure.

If we proceed in this manner from the key-block at the crossing down one rib, taking account of the load on the rib, we may find the line of resistance just as for a simple arch and compute bending moment and shearing just as before.

It should be noted that the geometric form of the rib necessarily leads to a weak arch, so that special care must be taken in providing sufficient depth. From this case we can easily proceed to the discussion of Gothic vaulting.

Before, however, this is considered the question of the line of resistance in a pointed arch needs brief notice

It is of course obvious that the line of resistance must follow a curve closely resembling that of the arch itself, but it should further be noticed that, since there is no keystone but a vertical middle joint, the crown thrust must be absolutely horizontal, *i.e.*, the loading must be exactly symmetrical. Subject to this proviso, the methods employed for the ordinary arch will be applicable to this case also.

It will be remembered that the panelling blocks have their transverse joints approximately perpendicular to a line bisecting the angle made by the wall ribs with the diagonal ribs. Lierne ribs may be regarded as merely serving to stiffen the panelling.

The order of procedure is then as follows, for a level ridge vault:

(1) Find the line of resistance in each of the side arches, and notice if these are of themselves sufficiently deep.

(2) Take the skewback thrusts from the intersecting vaults and combine them into a series of resultants acting through the diagonal ribs, thus finding lines of resistance for the latter.

(3) Notice that the panelling joints are nowhere inclined to more than the angle of friction with the diagonal or side arches.

As an alternative to this last, a vertical section may be taken through the panelling, bisecting the angle

between a side arch and the diagonal rib, and the loads on unit width studied. If the line of resistance passes outside the middle third, then the panelling should be deepened.

It is a not uncommon practice to fill in the vaulting from above with concrete, deep over the arch springings but thin over the crown. This steepens the lines of resistance and also deepens the arch, thus doubly contributing strength.

Domical vaults may be treated similarly, save that lines of resistance need to be drawn for the apex arches, and it should be noticed whether the skewback thrusts from the same are sufficiently great or oblique in direction to push out the crown voussoirs of the side arches.

The proportions commonly adopted for Gothic vaulting are ample, and if a concrete filling be employed there is no doubt that the usual forms of construction are sufficiently strong without special design being necessary.

Skew Arches

It is well understood that the skew, helical, or spiral construction for oblique arches is intended to give the necessary strength to the arch. This object is achieved in two directions :

(1) The coursing and transverse joints are by the use of helical lines made perpendicular to each other.

(2) The transverse joints are by the helical arrangement prevented from breaking out on the face of the arch and so leaving a part of the work unsupported. In studying this type then from the point of view of strength, it will be convenient to imagine the heading joints to be unbroken, so that the auxiliary effect of bonding along the coursing spirals is neglected, or, rather, taken as an additional security.

It will be convenient to give symbols to the three angles made by any one course at a certain point.

(1) θ = angle between the tangent to the coursing spiral and the horizon.

(2) ϕ = angle between the tangent to the coursing spiral and a line drawn perpendicular to the faces of the arch.

(3) ψ = angle between the joint and the perpendicular, *i.e.*, the angle between a perpendicular to the coursing spiral, tangential to the arch, and the horizontal.

Further, let us assume a weight W on the crown block or keystone between any pair of heading spirals (say unit distance apart).

We can then proceed from block to block in the following manner:

On the keystone of any one ring (included between the above-mentioned pair of heading spirals) θ is zero, *i.e.*, the coursing spiral is there horizontal in direction, so that the thrusts on the adjacent joints are connected by the expression

$$W_1 = 2T \sin\frac{\psi}{2}$$

$$\text{or} \quad T = \frac{W_1}{2 \sin\frac{\psi}{2}} \qquad (3)$$

This thrust is in a plane inclined ϕ from a sectional plane perpendicular to the faces of the arch, *i.e.*, in the tangent plane to the heading spirals.

As we come to the second block we have to remember that the bed-joint of this block lies in a new coursing spiral ϕ_1 from the perpendicular section, and the thrust T so far as that block is concerned must be resolved into two components, $T\cos(\phi - \phi_1)$ and $T\sin(\phi - \phi_1)$, the first perpendicular to that joint and the second tangential (shearing).

Furthermore, we have to recognise that the coursing spiral of the second joint is also no longer horizontal, but inclined at a small angle θ_2 to the horizon. Hence the load W_2 in the second block must be regarded as being supported by two upward reactions $W_2 \cos\theta_2$ and $W_2 \sin\theta_2$, the first being the resultant of the thrust from the crown block and the third block in the ring

being studied, and the second ($W_2 \sin \theta_2$) being laterally transmitted and tending to cause shear.

Hence we have approximately a shearing force

$$S = W_2 \sin \theta_2 + T \sin (\phi - \phi_1) \qquad (4)$$

and a thrust on to the third block which is the resultant of $T \cos (\phi - \phi_1)$ and $W_2 \cos \theta_2$; that is,

$$T_2 = T \cos (\phi - \phi_1)_{\psi 2} + W_2 \cos \theta_{2\pi/2} \qquad (5)$$

where T_2 is the thrust on the third block, ψ_2 is the angle between a perpendicular to the second joint (*i.e.*, the face of the third block). It should be observed that this is a vector summation. An algebraical formula can easily be constructed, but it is preferable to do the work graphically.

Similarly we may proceed to the fourth block. The load W_3 on the third block must be split into components $W_3 \cos \theta_3$ and $W_3 \sin \theta_3$, the former being that affecting the ring, and T_2 must be split into components $T_2 \cos (\phi_1 - \phi_2)$ and $T_2 \sin (\phi_1 - \phi_2)$, of which the first is the important one in studying the ring. The thrust on the fourth block will then be by vector summation

$$T_3 = T_2 \cos (\phi_1 - \phi_2)_{\psi 3} + W_3 \cos \theta_{3\pi/2}.$$

Proceeding thus, a line of resistance can be drawn for the whole of the ring, and if a section following the heading spirals be drawn on a base equal to the diagonal

span, it will at once be seen whether or not dangerous bending effects are likely to occur.

It will, of course, be remembered that the skewbacks for skew arches must be cut to the coursing spirals so that the shear component will cause the resultant thrust to be as nearly as practicable perpendicular to the same.

In connection with arched ribs it may be useful to notice the theoretical forms of arch which are the most suitable for certain types of loading.

Two forms are principally noteworthy :

(1) *Arch with uniformly distributed load* (*i.e.*, constant load per unit span).

It is well known from the principles of bending moment that a beam whose depth varies inversely as the square of the distance from the ends (*i.e.*, parabolic) is the most economical of material. Similarly it will be obvious that the link polygon for this kind of load (which would be a diagram of bending moment on a beam) will be parabolic in form ; *i.e.*, if y be the depth below the crown of the figure, the horizontal distance from the centre line to either side of the link polygon will be equal to x, so that $\kappa x^2 = y$.

If now we write $2x_1$ as the distance between the inner edges of the middle third at springing level (*i.e.*, the span and twice the horizontal distance from the intrados to the middle third) and make y_1 = vertical height of

the arch from the upper edge of the middle third to the springing level, we may write

$$y_1 = \kappa x_1^2$$

and

$$\kappa = \frac{y_1}{x_1^2},$$

and we can calculate the position of the line of resistance as compared with the centre line of the arch by subtract-

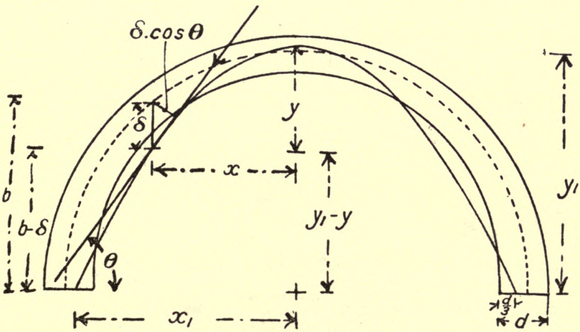

FIG. 14.

ing from the height of the latter above springing level the value

$$u_1 - y = y_1 - \kappa x^2,$$

where x is the horizontal distance from the centre line; so that if the centre line of the arch is b above the springing line the line of resistance is situated at a distance vertically below equal to $\delta = b - (y_1 - \kappa x^2)$ (Fig. 14).

Now in the previous chapter we have seen that there is the same horizontal component of the thrust everywhere. If this be H it is obvious that the moment on the section of the arch at the place considered is

$$H\delta = H\{b - (y_1 \kappa x^2)\}.$$

Furthermore, if the line of resistance is inclined θ to the horizontal at the place considered, the actual thrust in the direction tangential to the line of resistance is H sec θ.

H can be found by drawing a polar diagram as before.

[NOTE.—This may be employed to prove the truth of the above formula for the bending moment. The line of thrust passes below the centre line at a *perpendicular* distance $\delta \cos \theta$, but $H \sec \theta \cdot \delta \cos \theta = H\delta$, as above.]

Any case which has a definite distribution of the load may be similarly dealt with when it is remembered that the link polygon for that system of loading is not only a type of the line of resistance, but is also a diagram of the bending moments on a beam similarly loaded, so that when the latter are known, and certain positions can be predicated for the line

of resistance, this line can be put on to the arch at once.*

(2) *Arch with uniformly distributed load along its curve* (*i.e.*, constant load per unit length).

In this case the curve is one known as the "catenary," and we have the following awkward relation between the distance from the centre line x, and the depth from the crown to the curve y:

$$y = c \cosh \frac{x}{c} \text{ or } \frac{y}{c} = \cosh \frac{x}{c},$$

where $\cosh\left(\frac{x}{c}\right)$ represents a mathematical quantity, tables of which are to be found in most engineering books.† The following values may be useful:

$\frac{x}{c}$	$\frac{y}{c}$	$\frac{x}{c}$	$\frac{y}{c}$
·0	1·0	2·5	6·132
0·5	1·128	3·0	10·07
1·0	1·543	4·0	27·31
1·5	2·352	5·0	74·21
2·0	3·762	6·0	201·72

* All arch problems really reduce themselves to this. A link polygon for the loads being fitted through three points, its deviation, from the centre line anywhere is calculable (graphically or otherwise), and the latter is proportional to the moment on the arch at that point.

† Cosh $\frac{x}{c} = \frac{1}{2}(E^{x/c} = E^{-x/c})$, so that mathematical readers can easily calculate it.

The constant $c = \frac{H}{w}$, where H is the constant horizontal thrust, and w the load per foot of girth. Since c is required before the curve can be drawn, H must be calculated, as may be done easily in the following manner.

Assume the loads to be spaced at convenient short intervals of girth (say $\frac{1}{2}$ ft.), and calculate the total moment of these about the inner edge of the middle third at the springing level, for *half* the arch. Let the total be M, then

$$M = Hy_1,$$

where y_1 is as given before, viz., the height from the springing level to the crown of the middle third.

Thus we have

$$c = \frac{H}{w} = \frac{M}{y_1 w}.$$

Divide this into x, the horizontal distance from the centre line to any point under consideration, and find $\frac{y}{c}$ from the table, converting the units if the measurements do not work out conveniently. Multiply this by c, and we have y, the distance of the catenary below the crown level of the middle third.

Then, as before,

$$\delta = b - (y_1 - y)$$

$$= b - \left(c \cosh \frac{x_1}{c} - \cosh \frac{x}{c} \right),$$

where x_1 is the distance from centre line to the inner edge of the middle third at springing level, and the moment on the section is $H\delta$.

These two cases are sometimes convenient in calculating the line of resistance. It should be noticed that the method is absolutely identical with the graphical one previously employed.

CHAPTER VII

DOMES

THE balance of forces within a dome is very similar to that which occurs in an arch, and as a matter of fact the forces in any vertical section may be regarded as identical with those in an arch of the same form provided that the lateral pressures have already been considered.

If the keystone is surrounded by n voussoirs (Fig. 15) we may regard each as receiving an equal thrust whose value is approximately $\frac{W}{n}$ cosec θ, when W is the load on the keystone, θ the inclination of the bed-joint to the vertical, or rather the angle between the axis of the dome and the conical surface of the joint.

The stability of any other voussoirs may be regarded conveniently in the following manner, on the assumption that the blocks are uncemented and frictionless.

Taking a vertical section, we find the weight of the voussoir (one load on it, if any) acting vertically down-

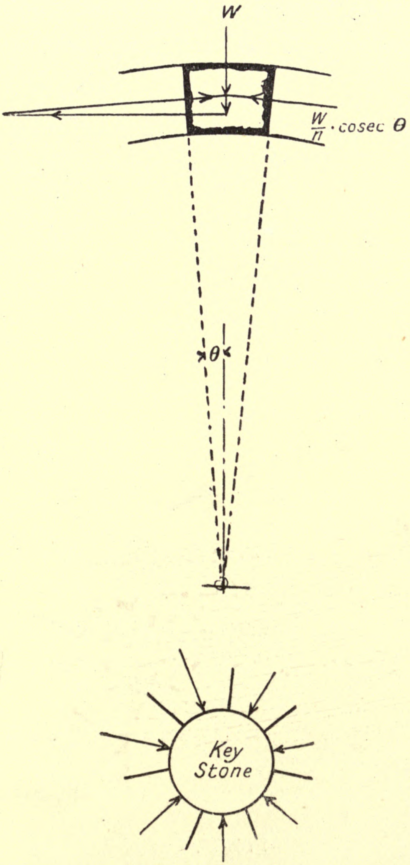

FIG. 15.

wards through the centre of gravity of the block, and the thrust from the next block above. If the resultant

of these two is perpendicular to the lower bed-joint there is no tendency to slip. If it incline inwards, then the

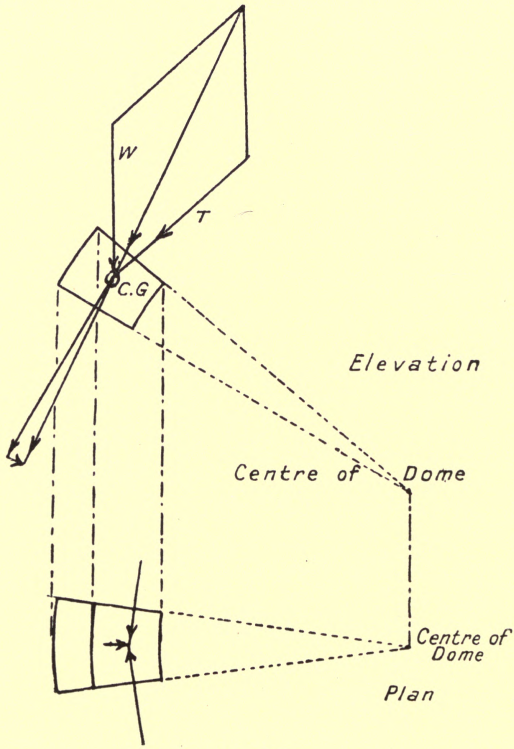

FIG. 16.

component parallel to the bed-joints tends to push the voussoir inwards, and this may be regarded as resisted by the lateral resistances acting normally to the

vertical planes of the heading-joints. If the resultant is directed outwards, shearing and friction must provide the necessary resistance (Fig. 16).

We may thus consider the dome in the following manner, working from the key-block. Let the load on this be W_0, and in the first ring let there be n_1 blocks. The thrust from the key-block on to each of these blocks is

$$T_1 = \frac{W_0}{n_1} \operatorname{cosec} \theta_1 \qquad (1)$$

If the load on each of these is W_1, then the vertica component of the resultant is

$$W_1 + \frac{W_0}{n_1} \operatorname{cosec} \theta_1 \,.\, \sin \theta_1$$

$$= W_1 + \frac{W_0}{n_1} \qquad (2)$$

and the horizontal component is

$$\frac{W_0}{n_1} \operatorname{cosec} \theta \,.\, \cos \theta_1 = \frac{W_0}{n_1} \cot \theta_1 \qquad (3)$$

If the bed-joints of the first ring are inclined θ_2 to the vertical, the component thrust parallel to the joint is

$$F_1 = \left(W_1 + \frac{W_0}{n_1}\right) \cos \theta_2 - \left(\frac{W_0}{n_1} \cot \theta_1\right) \sin \theta_2 \qquad (4)$$

and normal to the joint

$$T_2 = \left(W_1 + \frac{W_0}{n_1}\right) \sin \theta_2 + \left(\frac{W_0}{n_1} \cot \theta_1\right) \cos \theta_2 \qquad (5)$$

The latter thrust is transmitted to the second ring, and if there are n_2 blocks in the latter, each may be regarded as subject to a thrust

$$= \frac{n_1}{n_2} T_2 \qquad (6)$$

If F_1 is positive, then there is an inward thrust of this magnitude, and since there are n_1 blocks in the first ring the vertical planes of the joints between these planes are inclined to one another at an angle $= \frac{2\pi}{n_1}$ and the thrust S_1 across the faces of these vertical joints.

$$S_1 = \frac{F_1}{2 \sin\left(\frac{1}{2} \cdot \frac{2\pi}{n_1}\right)} = \frac{F_1}{2 \sin \cdot \frac{\pi}{n_1}} \qquad (7)$$

So we may proceed from ring to ring, substituting successively

n_2 for n_1, n_3 for n_2, &c.,

T_2 for T_1, T_3 for T_2, &c.,

and similarly for S_1, F_1, θ_1, and W_1, noting of course that the resultants will each time combine with the weights on the successive voussoirs.

If at any joint the quantity F_1 (the tangential component of the resultant from above) becomes negative, then shearing and friction must be considered as

already mentioned. Thus if F is negative we may write

$$F = f_s A + \mu T \qquad (8)$$

when f_s is the shearing stress per sq. ft. of section A, μ is the coefficient of friction, and T the normal force as before. From the above formula

$$f_s = \frac{F - \mu T}{A} \qquad (8a)$$

(μ is about ·7. *See* chap. i.)

The bending moment on the joint is found by drawing the resultant of W_n and T_n for any one block and noticing where it cuts the bed-joint. If it passes through a point δ distance from the centre the moment

$$M = T_{n1} \times \delta \qquad (9)$$

when T_{n+1} stands for the normal component of the resultant of W_n and T_n.

The same method is applicable to domes of any vertical section, provided they are circular in plan. If a dome is not circular in plan, allowance for the variation in the joints must be made when applying (7). It will be found that there is a tendency in such a dome to burst outwards at the sides or the ends, according as the line of resistance falls.

In some cases domes which are only visible externally

are tied across so as to prevent the horizontal thrust being transmitted to the lower work. The pull in each such tie-rod will be

$$P = \frac{n_1}{N} T_1 \cos \theta_1 \qquad (10)$$

where n_1, θ, and T_1 are obtained from (1) and N is the number of rods.

Generally, however, the dome is supported on a sub-dome or pendentive, which again rests on walls or piers. If the latter there are generally four (or more) in number.

The pendentive dome generally consists only of four spherical triangles, combining to form a complete circle under the true dome, and the lower cusps or angles descending to the four piers, arches being sprung between the piers.*

If a section be taken across the dome centrally and parallel to a line connecting the two piers, the arch of the dome will be seen to spring from the crown of the arch between the piers Another section taken diagonally across two alternate piers will show the dome springing from the rim of the pendentive and the latter sending its thrusts down to the piers. A third view taken externally shows the inter-pier arches sending

* See Mitchell's *Building Construction*, Fletcher's *Architecture*, or any text-book on Byzantine architecture or masonry.

down their thrusts to the piers, and a plan will show that all these thrusts tend to push the piers diagonally outwards.

The total value of this diagonal thrust on each pier is

$$T_P = \frac{\Sigma(W)}{4} \tan \theta_P = \frac{n_1}{4} T_1 \cdot \cos \theta_1 \qquad (11)$$

where T_P is the diagonal horizontal thrust in lbs., $\Sigma(W)$ is the total load on the dome in lbs., and θ_1 the angle made with the vertical by the thrust from the pendentive, or n_1, T, and θ_1 are as in (1) and (10).

If the pier is square, S feet long on each face, then the moment of inertia of the section about a diagonal axis is

$$\frac{S^4}{12} \qquad (12)$$

and the maximum stress due to bending is

$$f = \frac{12 T_P H}{\sqrt{2}\, S^3} \qquad (13)$$

where f is the maximum stress (lbs. per sq. ft.), T_P is the diagonal thrust from (11) in lbs., H is the height of the pier in feet and S the width of the face. The dead load on each pier is $\Sigma(w) \div 4$, so that the actual stresses are

$$\frac{\Sigma(w)}{4S^2} \pm \frac{3 n_1 T_1 \cos \theta_1 \cdot H}{\sqrt{2}\, S^3} \qquad (14)$$

The loads on the inter-pier arches are practically uniformly distributed, so that no great difficulty arises with them.

The thrust from the super-dome to the pendentive may be regarded as the initial thrust on the latter, and may be continued down the rings. Each ring, however, consists of a smaller number of blocks, and the thrust must be increased from ring to ring in inverse ratio to the number of blocks until the supporting piers are reached.

Very frequently in domed construction, particularly in the Renaissance style, heavy finial ornaments or towers are built above the dome, and must be considered in its design. The effect of their weight is threefold:

(1) To steepen the diagram;
(2) To increase the horizontal thrust;
(3) To increase correspondingly all the thrusts.

The first item is of considerable importance, since we find that many of the domes, if hemispherical, are subject to considerable bending, so that many architects have found it advisable to employ hyberbolical forms. Thus in St. Paul's Cathedral the true (structural) dome is conical with a slightly curved apex. If, as in the case mentioned, this dome is not sightly it has to be screened with false light domes. In some cases (as the Brompton

Oratory) a suitable visible dome may be so formed, something after the fashion of a Saracenic cupola.

If the reader draws some lines of resistance on which the central load is great, it will be easily seen why such a device is necessary, the only alternative being to thicken the dome, with a consequent increase of immediate load, and also greater effects on the substructure.

Similarly it will be found that elliptic domes (*i.e.*, elliptic in any vertical section) are most suitable when the loading is greatest at parts most remote from the apex.

Ogee domes, such as occur in Saracenic work, are not capable of supporting great loads, since the line of resistance in such a case will necessarily leave the middle third at some point.

In cases where there is any doubt as to the security of an existing dome, or where it seems desirable to take special precautions to prevent collapse, iron bands are frequently put round the haunch rings. These serve the double purpose of reinforcing the rings to resist tension, and also to balance the effect of an outward-acting shearing force.

If the shearing force be computed by formula (7) for a joint in any particular ring we find the total outward force in the ring is nS, where n is the number of joints

and S the shearing on one joint. If this be divided by π, we have the diametric effect (as in boiler design), so that we may write

$$2f_t a = \frac{nS}{\pi} \qquad (15)$$

where f_t is the tension per sq. in. in the band and a its sectional area.

As far as bending effect is concerned, it must be realised that horizontal rings are of little use in resisting the bending effects of the vertical lines of resistance, but will be useful if, owing to the irregular distribution of the load or the special form of the dome, the lines of resistance running horizontally round each ring produce bending. If the iron be *not* fixed into the stone, it reinforces the latter by the total amount of its tension, so that it is thereby able to resist bending to an extent $f_t a \times \delta$, where δ is the distance to the centre line of the ring. If the iron is cemented into the stone the tensile resistance is less but the compressive resistance is greater.

It has frequently been remarked by architects and engineers who have visited the Orient that the domes so frequently erected there are, according to our standards, unstable and yet rarely seem to fail. It will, then, perhaps, be useful to point out wherein the theory fails to take into account practical safeguards.

If we look through the rules already given it will be noticed that the following assumptions underlie them:

(1) That masonry should not be subject to more than one-tenth its ultimate stress.

This alone would account for many cases in which failure does not occur, for undoubtedly, if we could prophesy an absence of vibration and flaws in the stone and joints, a much lower factor of safety could be used.

(2) That the line of resistance should fall within the middle third of the joint.

This condition ensures that tension shall not occur in the work. On the other hand it is quite possible that in many cases tension might safely occur up to a certain limit, so that this again will explain the permanence of domes built without such precautions being taken.

(3) That the resistance of the joints is uniform or uniformly varying.

This is a condition which must undoubtedly be assumed in constructing a working theory, although there is every probability that it is not realised in fact. Irregularities in the mortar and on the stone will frequently cause irregular resistance, and, moreover, recent researches on the subject of reinforced concrete have indirectly shown that masonry is not by any means uniformly elastic.

Hence we may conclude that in many cases the nature of the construction (guided by experience inexpressible in words) has been such that the extra resistance obtainable in this way has been taken advantage of.

Another point which perhaps deserves attention is the statement commonly made that large domes, such as those of St. Sophia or St. Peter (Rome) or of Florence Cathedral, have been built without this theory. This is, of course, the old question of practical instinct *versus* theory, and it seems scarcely necessary to point out that the theory is but an outcome of the results obtained by long practice, and that very probably it would be found that now such domes could be constructed with even greater economy of material than was displayed by the illustrious architects who executed these works.

Before leaving the subject of domes, it will be useful to refer to the question of piercings and lanterns. It is obvious that any such opening in a dome must be surrounded with an arch ring, which serves the same purpose as the masonry occupying the same space would do in a complete dome.

Thus a lantern on the apex of a dome is surrounded with a horizontal ring, which forms the key-block to the dome. This ring acts in precisely the same manner

as the key-block would have done, and must be designed to resist the same forces.

Similarly any piercings in the sides of a dome will take the place of the masonry whose room it occupies and be subject to the same forces. One particular point in this connection may be mentioned, viz., that since the thrust increases as we descend the dome, the lower part of the ring round an opening will happen to be subject to greater forces than the upper. On the other hand, the greater size of the rings in the lower parts of the dome will distribute the pressures over a larger surface, so that it may be doubted whether there any actual increase, and in small domes it will certainly suffice to construct an arch ring of uniform strength round the opening. Very frequently stilted arches are used (compare the lights round the base of the main dome at St. Sophia), and the lower blocks in such cases will be subject to considerable horizontal shearing from the dome ring abutting against them. The direct loading on to the arch will, however, tend to increase the frictional resistance, so that rarely will it be necessary to use special construction to resist the extra force.

In conclusion it may be pointed out that the dome is a construction which gives the maximum amount of uniform distribution of pressure.

CHAPTER VIII

RETAINING WALLS AND DAMS

THE question of walls required to resist lateral pressure has received considerable attention. It involves two problems, both of which are, to a certain extent, indeterminate, viz.:

(1) Lateral pressure of retained material.

(2) Stresses in a wall subject to such lateral pressure.

For completeness' sake we may devote a little attention to the former problem, but seeing that the latter is more important as regards the masonry itself, this will be first considered, and the lateral forces assumed both as to magnitude and position.

It is essential that all the forces acting on the wall, taken as a whole, shall be in equilibrium, unless the wall is rigidly secured to the earth, so that it may be regarded as an integral portion of the same. The latter assumption is only made under exceptional circumstances, and it is usual to simply assume that:

(1) The moment of the lateral force is balanced by

the moment of the weight of the wall about some point between the centre of gravity (projected on to the base) and the outer edge of the base.

(2) The shearing effect of the lateral force is balanced by the frictional resistance of the blocks to sliding.

In other words, the wall is regarded as consisting only of uncemented blocks. The margin of safety so secured is perhaps, in some cases, excessive, but this point will be considered later.

The stresses in the material of the wall are of three kinds: (*a*) bending, (*b*) compression, (*c*) shearing.

The first is due to the turning moment acting on the wall, the second to the weight of the wall, and the third to the sliding effect of the lateral forces.

In any mass of masonry the pressure on the base is as nearly as possible represented by the rule

$$\text{Pressure (lbs. per sq. ft.)} = \frac{\text{weight (lbs.)}}{\text{area of base (sq. ft.)}}.$$

This is, if the wall is thoroughly bonded, true even when the faces are battered.

It is convenient to consider only one foot-run of the wall, for if that piece is stable all similar pieces under the same conditions will also be stable.

Hence we may write

$$P = \frac{W}{D} \tag{1}$$

where P is the compressive stress at any level in lbs. per sq. ft., w is the load in lbs. above that level, and D is the diameter of the wall in feet at that level.

The lateral force, which for the present we will call F lbs., will generally be horizontal in action (Dr. Scheffler opines that in a surcharged retaining wall it acts parallel to the slope of surcharge, but Rankine adopts the view that the action is horizontal. The latter is on the safe side), and if its height above the level of the section considered is h ft., then the turning moment on this section is

$$M = Fh \qquad (2)$$

It is shown in works on applied mechanics that the turning moment M, the dimensions of a rectangular section, and the maximum stress are connected by the following rule:

$$M = f\frac{BD^2}{6} \qquad (3)$$

B is unity (one foot run of the wall), so that

$$f = \frac{6M}{D^2} \qquad (4)$$

This f is the stress in the masonry due to the latera pressure, on the outside of the wall compressive and on the inside tensile, so that, combining it with the dead pressure, we have two useful formulæ:

$$\left.\begin{aligned}\text{Maximum compression} &= \frac{W}{D} + \frac{6M}{D^2}\\ \text{Minimum compression} &= \frac{W}{D} - \frac{6M}{D^2}\\ \text{(or tension)} &\end{aligned}\right\} \quad (5)$$

Since the blocks are regarded as uncemented it is obviously undesirable that the masonry should be in

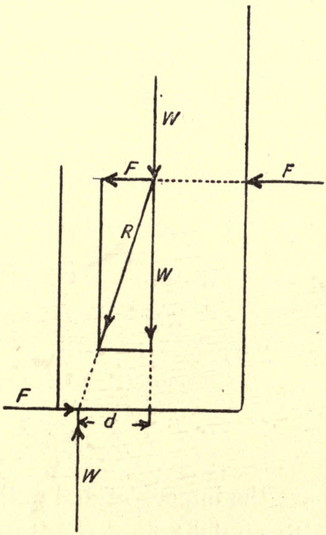

FIG. 17.

tension, so that if we write the second expression as equal to zero we have the case where the tension due to bending just neutralises the compression due to weight. It will be convenient to know where, under these conditions,

the "resultant force" passes through the section. The "resultant force" here referred to means the resultant of the total load above the section, and the lateral pressure acting on the wall above the section. Thus, if AB is the section, and F and W act in the positions shown (Fig. 17), the resultant, R, passes through a point, d, from the centre of the section. Splitting this resultant into the original components, we see that it may be regarded as a force, W, acting vertically on the section at d from the centre, and a shearing force on the section equal to F. The moment of the first about the centre $= \mathrm{W}d$, so that we have

$$\mathrm{M} = \mathrm{W}d = \mathrm{F}h \qquad (2a)$$

Putting this value for M in (5), we have

$$\frac{\mathrm{W}}{\mathrm{D}} = \frac{6\mathrm{W}d}{\mathrm{D}^2} = 0;$$

so that

$$d = \frac{\mathrm{D}}{6} \qquad (6)$$

Hence we have the important and well-known rule:

In a wall with straight faces (*i.e.*, in plan), in order that there shall be no tension at the interior edge of a section, the resultant of the forces acting on the wall above that section must pass within the middle third of the base, *i.e.*, not more than one-sixth the width of the section in front of the centre of gravity.

In cases, by no means few in number, where the above rules lead to very thick walls,* various devices are employed to economise masonry. One of the simplest is to incline the wall backwards against the retained material. The effect is to throw the centre of gravity backwards, so that the resultant is also brought back. Another common method is to make the wall thicker at the base, so that D is increased there (and *d* proportionately). The faces may be "battered," *i.e.*, evenly sloped or stepped. If the wall is stepped on the back, the earth on the steps can be included in the weight of the wall.

Yet another method is to buttress the wall at intervals with short pieces of thick walling, so that the mean value of D is again increased. The buttresses may be outside or inside. If inside, the wall is sprung in arches between, the construction being termed "counterforted."

The wall may also be curved on the outer (and sometimes also the inner) face, so that the thickness increases more rapidly than the depth. This arrange-

* Putting (2*a*) in the first rule of (5), we have

$$\text{maximum compression} = \frac{2W}{D}.$$

and since at the bottom of the wall $W = DHw$, where H is the height and w the weight per cubic foot, the connection between the height allowable and the maximum compression is :

$$\text{maximum compression} = 2Hw.$$

ment is often employed for dams where the total water pressure varies as the square of the depth, and is unaffected by friction. It will, however, be more convenient to deal with this matter when the nature of the lateral pressure has been considered.

Lateral pressure on a wall may arise from a number of causes, which may, however, be specified as follows:

(1) *Water pressures.*
(2) *Earth pressures.*
(3) *Constructional pressures.*

The last case, including flying buttresses, oblique struts, arch abutments, &c., may conveniently be considered as a variation of the arch problem.

Water pressure has certain very simple characteristics.

(*a*) It acts perpendicularly to all surfaces opposed to it.

(*b*) Its magnitude *at any point* is proportional to the depth of water.

(*c*) Its *total* magnitude down to any point is proportional to the square of the depth.

(*d*) It depends entirely on depth and is irrespective of quantity. Thus two dams $\frac{1}{4}''$ apart, the intervening space being filled with water, are subject to the same pressure as the retaining walls or banks of a reservoir

containing many million tons, if the heights are the same. (Moral—Beware of cracks in dams.)

(*e*) The *mean* pressure of water is at half the depth.

(*f*) The resultant pressure of water on a plane and continuous surface (as that of a battered or perpendicular wall) is two-thirds the depth from the surface.

The standard rule referred to in (*c*) is as follows :

$$F = \tfrac{1}{2} = w_0 H^2 = 31{\cdot}25 H^2 \qquad (7)$$

where w_0 is the weight of a cubic foot of water, and H is the depth in feet, F the total pressure in lbs. on a vertical and plane surface extending the full depth.

If the surface is not vertical, but of length H sec θ, where θ is its inclination with the vertical, the total pressure is

$$31{\cdot}25 H^2 \sec\theta \qquad (7a)$$

The moment of the first force is

$$Fh = \frac{FH}{3} = 10{\cdot}41 H^3 \qquad (8)$$

The moment of the force on the inclined wall is

$$10{\cdot}41 H^3 \sec^2\theta \qquad (8a)$$

It should be noted that the latter force acts at an angle θ with the horizontal, and that the moment is expressed about the *foot of the internal slope*, not about

the centre of the wall, as is required for calculating the stresses. The latter is as follows:

$$10\cdot41H^3 \sec^2 \theta - 15\cdot62H^2D \sin \theta \qquad (8b)$$

where D is the base thickness (horizontal).

In the case where the wall has an irregular or curved internal face the matter must be treated in rather a different manner. Two systems may be adopted. The first is more in conformity with the preceding method, and is as follows:

Take the face bit by bit as it changes slope (or if it is a curve assume it made up of a number of chords), and find the pressure on each bit as follows:

Let the depth at one end of the length be H_1 and at the other H_2. Then the mean pressure is

$$31\cdot25\,(H_1 + H_2),$$

and if the length is L then the total pressure is

$$31\cdot25L\,(H_1 + H_2).$$

If L is inclined θ from the perpendicular, then

$$L = (H_2 - H_1) \sec \theta,$$

so that the total pressure AF

$$= 31\cdot25\,(H_2^2 - H_1^2) \sec \theta_1 \qquad (9)$$

Measure the distance perpendicularly from the line of action of this pressure (inclined θ_1 to the horizon) to the centre of the wall at the section considered, and we

obtain the moment of the force about that point. The sum of all such moments taken above the section will give M, and the wall may be designed to accord with the previous rules. If L in the section taken is small, the pressure may be assumed to act through the middle of it. If it is comparatively large the following simple rule may be used to find the centre of pressure (*i.e.*, the point at which the resultant force acts):

Draw at each end of L a perpendicular proportionate to H_1 and H_2. Join the heads of these lines and find the centre of gravity of the enclosed trapezium. The line of action will pass through this centre of gravity and be perpendicular to L.

Instead of taking the moments separately, the various forces may be combined by a link polygon so as to find the resultant force. This resultant will then correspond to F in the previous problems, and, being multiplied into its perpendicular distance in feet from the centre of the wall at the section considered, will give the required moment M. Notice that no forces acting below the section may be included.

The second method is to take the wall bit by bit as already suggested, finding the resultant force on each bit, and then combine this with the resultant force coming from the section above. Thus in the first section there will be the weight of that section of the

wall and the lateral pressure on that section. The resultant of these will be continued to combine with the weight of the second section (weights, of course, will act through the centres of gravity of each section), and the common resultant will then be combined with the lateral pressure on the second section, and so on until the bottom of the wall is reached. In this manner a line of thrust similar to that employed in arches will be mapped out. Its intersection with a section taken at any level will indicate by its position on the section and its components the direct bending and shearing forces as in the case of the arch.

(This same method applies to lateral buttresses.)

Earth Pressures.—These are identical in kind with water pressure, but less in magnitude where regard is had to the actual weight of the supported material.

Referring back to rule (7), we have $F = \frac{1}{2}w_o H^2$, where w_o is the weight of a cubic foot of water (62·5 lbs. nearly). The same rule applies here, but we must multiply by some fraction less than unity, so as to allow for the reduction of pressure due to internal friction; thus we have

$$F_e = \frac{1}{2}w_e H^2 \times \kappa \qquad (10)$$

When w_e is the weight of a cubic foot of retained material (sand 100 to 170, according to wetness, shingle

90, clay 120), κ depends on the nature of the material, chemically and physically. In experiments on friction we find that the angle to which a slope may be fixed without a block on it slipping is a convenient measure of the friction between the block and the slope. Similarly, a mass of loose material slides on itself until the slope has a certain angle, which is analogous, if not identical, with the one in the experiment. This angle is termed the angle of repose (Rankine), and generally denoted by the Greek letter ϕ (phi). In the case of a wall supporting a bank with a horizontal surface

$$\kappa = \frac{1 - \sin \phi}{1 + \sin \phi} \qquad (11)$$

surcharged to the angle ϕ; $\kappa = \cos \phi$ (11a)

The angle of repose varies from 0° (water) to 90° (hard rock). Important intermediate values are as follows:

Wet clay, 16°.
Sand, 22°.
Shingle, 40°.
Well-drained clay or compact earth, 45°.

When $\phi = 42°$, κ in (11) is about $\frac{1}{5}$, and in (11a) about $\frac{3}{4}$.

In cases where the slope of surcharge is less than

the angle of repose, or becomes more complex, being expressed by

$$\kappa = \cos\beta \frac{\cos\beta - \sqrt{\cos^2\beta - \cos^2\phi}}{\cos\beta + \sqrt{\cos^2\beta - \cos^2\phi}} \quad (11b)$$

(β is angle of surcharge).

This somewhat cumbrous expression will rarely need to be used, since the slope of surcharge is usually the angle of repose.

The resultant force acts, as in the case of water, at two-thirds the depth of the supported mass.

Another theory of earth pressure, which is conveniently adapted to graphical methods and differs but slightly in its results from the foregoing, deduces the conclusion that of the mass retained a certain wedge alone need be considered as producing lateral pressure. This wedge is bounded by three planes:

(1) *The back of the wall.*

(2) *The surface of the surcharge.*

(3) *A plane making an angle* $= 45° - \frac{\phi}{2}$ *with the back of the wall.*

This mass is regarded as supported by the reactions from the wall (the lateral pressure), and reaction from the remaining earth and friction on the latter.

If the mass of the wedge be calculated and regarded

as acting through the centre of gravity, and resolved into two components, one horizontal and the other inclined at an angle ϕ upwards from the perpendicular

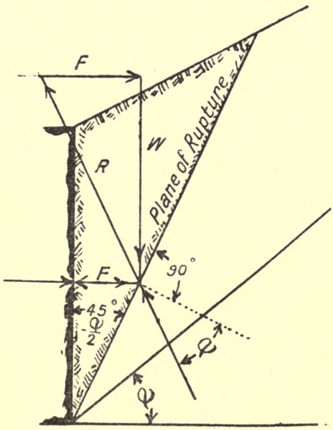

FIG. 18.

to the slope of the remaining mass, then the former is the lateral pressure.

This (Fig. 18) is perhaps the simplest method to apply, and is analytically correlated to the preceding. With regard to the shearing force acting through any section, it may perhaps be useful to point out that the usual assumption made is that the shearing force is distributed throughout the thickness in the same manner as in a simple rectangular beam—*i.e.*, two-thirds the mean value at the centre, zero at the edges, and a

parabolic variation between these points. Professor Karl Pearson has pointed out that the grounds for this assumption are uncertain, and a very interesting series of papers has appeared in the *Proceedings of the Institution of Civil Engineers* recently on the subject. It can hardly be said that a simple alternative has been arrived at, and as the matter will only be important in very large dams, it scarcely calls for mention here.

CHAPTER IX

ARTIFICIAL STONE AND CONCRETE

For the purpose of studying the stresses in masonry constructed with artificial materials it will be convenient to subdivide the latter in the following manner:

(*a*) *Terra-cotta blocks*, made hollow.

(*b*) *Silicated cement*, *concrete*, *or terra-cotta blocks*, cast solid.

(*c*) *Monolithic concrete.*

The first class is rarely required to bear any stresses greater than those produced by its own weight. If any case arises in which considerable stresses are produced it will be well to regard the resistance as wholly derivable from the terra-cotta shell, the Roman cement or other filling serving as a surplus resistance.

The second class differs, of course, but little from ordinary masonry. The tensile strength of the material has no peculiar properties.

The last form, however, needs special consideration for the following reasons:

(1) The absence of joints greatly increases the shearing and bending resistance.

(2) The monolithic character of the work ensures complete distribution of the load, and at the same time tends to reduce bending moments (*i.e.*, in cases such as floors fixed all round).

(3) A comparatively high tensile resistance may be resisted throughout a considerable length of work.

These effects may be summarised by saying that the continuous character of the structure leads to a similar continuity in the stresses and bending moments throughout the whole.

It is therefore the peculiar feature of monolithic structures to transmit their bending moments eventually to the ground, so that every individual member is subject to an external moment equal to its average impressed moment, the algebraic sum being zero.

Thus beams with central concentrated load W, span L, are subject to an impressed moment WL/4, but the ends being constrained to remain in their original direction there must be applied to those ends (and transmitted through the supporting walls) a moment

WL/8, being the average value of the impressed moment, $(WL/4 + 0) \div 2 = WL/8$, and the moment of any point in the beam is

$$M_o = M - WL/8 \qquad (1)$$

where M is the moment caused by the load on the beam, regarded as a case of simple support. At the faces of the walls the moment on the beam will be then $-WL/8$, and at a point $\frac{1}{4}$ span from the walls the moment is zero (the point of reflex curvature) and at the centre the moment is WL/8.

It will be obvious that the moment at any point is less than if the beams were simply supported.

In the case of a uniformly loaded beam, where the moment, if simply supported, is

$$M = \frac{wLx - wx^2}{2} \qquad (2)$$

where w is the load per foot run, L is the span, and x the distance from the abutment, the whole bending moment diagram is a parabola, whose vertex measurement is $wL^2/8$. Now the mean height of a parabolic segment is two-thirds the vertex height, so that the externally applied moment is $wL^2/12$, and the moment M_o at any point in the beam when part of a monolithic structure is

$$M_o = M - wL^2/12 \qquad (3)$$

or, substituting from above,

$$M_o = \frac{wLx - wx^2}{2} - \frac{wL^2}{12}.$$

This is, of course, zero when

$$6(Lx - x^2) = L^2,$$

so that

$$x = \frac{L}{2} \pm \frac{L}{\sqrt{12}} \qquad (4)$$

This means that the points of zero moment (reflex curvature) are respectively $L/\sqrt{12}$ on either side of the centre.

The supporting walls will of course have to transmit the constraining moment $wL^2/12$.

It will be noticed that in this case the maximum moment is not at the centre (where it has the value $wL^2/24$), but at the walls (where it is $wL^2/12$).

In the case of a beam with irregular loads a similar method may be employed. First calculate the moments as if the beam were simply supported, and then find the average moment. Subtract this, and the result is the moment when the ends are fixed. (Fig. 19.)

Thus in the case illustrated the moments under the two loads are respectively

$$\frac{W_1(b + c) + W_2c}{l}.a = M_1 \qquad (5)$$

and

$$\frac{W_1 a + W_2(a + b)}{l} . c = M_2 \qquad (6)$$

The mean value is found as follows:

$$\left(\frac{M_1}{2} . a + \frac{M_1 + M_2}{2} . b + \frac{M_2}{2} . c\right) \div l = M_m \qquad (7)$$

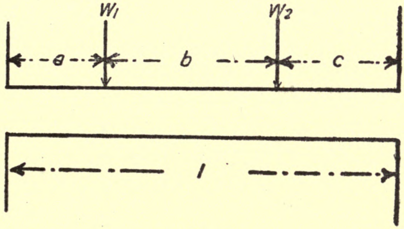

FIG. 19.

so that the real moments at the points under the loads are $M_1 - M_m$ and $M_2 - M_m$. At the ends the moment is M_m.

It is somewhat doubtful whether the principle of three moments may be applied to monolithic structures having continuous beams, because this principle implies that there is exactly equal settlement, and that the deflections everywhere are perfectly elastic. Recent researches as to ferro-concrete have demonstrated the fact that concrete has not a definite modulus of elasticity (although we often have to assume so), so that it

would appear inadvisable to apply the notion of perfect elasticity in such a severe case as this.

With regard to the vertical members of a monolithic structure, regard must be had to the manner in which they are connected to the horizontal ones. It has already been mentioned that the latter transmit to the vertical members certain moments as well as the direct load. If we disregard the crippling effect due to mere length, we may say that any vertical wall or column is subject to the following effects:

$$(1) \qquad W_c = \Sigma(W),$$

where $\Sigma(W)$ represents the sum of all the loads on it, including its own weight and the usual proportions of the floors, roof, live load, &c.;

$$(2) \qquad M_c = \Sigma(M),$$

where $\Sigma(M)$ stands for the *algebraic* sum of all the moments on it, from floors, transverse beams, roof frames, &c.; so that we may state the maximum and minimum stresses

$$f = \frac{W_c}{A} \pm \frac{M_c}{Z} \qquad (8)$$

where A is the area of the section considered (generally the base), and Z is the modulus of the same section.

By arranging floors, &c., systematically about the

vertical members, the algebraic sum of the moments may be reduced to a very small amount indeed.

In monolithic structures it will frequently happen that the floors are made self-supporting for a considerable area, *i.e.*, fixed at all edges. A panel of flooring of this kind tends to be subjected to stresses acting along lines radiating from its centre, but the exact magnitudes have not been settled. Rankine, Grashof, and others have deduced formulæ for calculating these stresses, but the whole matter is very uncertain. We may, however, assume that the material is able to resist bending effects in two directions at right angles to each other without interference. Let us first consider the case in which the panel is merely supported at all edges, not fixed. The total reaction on each edge may be supposed to be a quarter the total weight, although this assumes that all the supporting surfaces are mathematically level. Assuming the panel to be square, and the load per sq. ft. is w, the length of the sides being l, we have $wl^2/4$ as the reaction on each side. If we suppose this to be concentrated at the centre of each side, we shall have a moment at the centre where either transverse section is considered equal to $wl^2/4 \times (l/2 - l/4) = wl^2/16$ as a rough value.

Equating this to the moment of resistance of the transverse section, we get

$$f\frac{ld^2}{6}=\frac{wl^2}{16}$$

$$f=\frac{8wl}{3d^2} \qquad (9)$$

(d = thickness).

This is at the best an uncertain value, but it is probably higher than the real one, because we must recognise that the reactions on each side are actually distributed.

The following rules are commonly employed for these cases :

Circular Slab, supported all round, $f=\frac{5r^2w}{6d^2}$ (10)

where f is the maximum stress (tons per sq. ft.),
r is radius (ft.),
w is load (tons per sq. ft.),
d is depth (ft.).

Circular Slab, fixed all round, $f=\frac{2r^2w}{3d^2}$ (11)

Square Slab, fixed all round, $f=\frac{s^2w}{4d}$ (12)

where s is the length of the side (ft.).

Rectangular Slab, fixed all round, breadth b, length l,

$$f=\frac{l^4b^2w}{2d^2(l^4+b^4)} \qquad (13)$$

This last is the case which most frequently occurs, and applies to all coffer panels and floors of uniform thickness.

As an example, we will take the case of a panel forming part of a floor above a coffered ceiling, the breadth of coffer in the clear being 5 ft., length 6 ft., load per sq. ft. 200 lbs. (including floor), and thickness 1 ft.

$$f = \frac{6^4 \times 5^2 \times 200}{2 \times 1\,(6^4 + 5^4)}$$

$$= \frac{6,480,000}{3841} = \text{nearly } 1700 \text{ lbs. per sq. ft.}$$

It should be noticed that the tension is on the upper side near the edges, and on the under side in the central parts.

A continuous ground layer fixed under or into the walls is a similar case, and an example of this may be interesting, as showing the theoretical origin of the cracks which are not infrequently seen in such cases.

A floor is covered with a ground layer, the dimensions in the clear being 20 × 30 ft., and the load per sq. ft. (due to weight of the structure as a whole) say half a ton. The thickness of the ground layer may be taken at 18 ins.

$$f = \frac{30^4 \times 20^2 \times 1220}{2 \times (\frac{3}{2})^2 \times (30^4 + 20^4)},$$

$$= \frac{395{,}280{,}000{,}000}{4{,}365{,}000} = \text{nearly } 90{,}800 \text{ lbs.}$$

This stress no concrete could stand, so that we see the practice of putting a continuous ground layer in monolithic structures is not to be recommended without reinforcement. If a ground layer is employed at all, it should not be fixed to the main walls, but connected hermetically with an asphalt or other waterproof damp course, so that the actual weight of the building is borne by the foundations of the wall, and not by the concrete layer. In the latter case the concrete layer may be as thin as we please, provided the earth beneath is properly levelled, since it will be continuously supported and loaded only with its own weight.

This same rule may be employed to find the maximum length allowable in a panel, for by multiplying across the denominator we have

$$2d^2(l^4 + b^4)f = l^4b^2w\,;$$

so that

$$2d^2l^4 - l^4b^2w = -\,2d^2b^4f,$$

and

$$l^4(2d^2 - b^2w) = -\,2d^2b^4f,$$

or
$$- l^4 = \frac{2d^2b^4f}{2d^2 - b^2w}$$

$$l = b_4\sqrt[4]{\frac{2d^2f}{2d^2 - b^2w}} \qquad (14)$$

Thus if d be $1\frac{1}{4}$ ft., f 1 ton per sq. ft., w $\frac{1}{10}$ ton, and b 4 ft., we have

$$l = 4 \times \sqrt[4]{\frac{2(\frac{5}{4})^2 . 1}{2(\frac{5}{4})^2 - (4^2 \times \frac{1}{10})}}$$
$$= 4 \times \sqrt[4]{2{\cdot}05}$$
$$= 4 \times 1{\cdot}2 = 4{\cdot}8 \text{ ft.}$$

A point not usually noticed is that since slabs fixed at the edges are subject to a restraining moment at those edges, beams which hold the edges are subject to a twisting moment. It is obvious from the formula (13) that the bending moment (which is proportionate to the stress) is less than that of a beam supported at the two edges only in the ratio

$$\frac{l^4}{2(l^4 + b^4)},$$

so that to find the twisting moment on an edge we may take the moment $Wl/8$ as the value where two edges only are supported, W being the whole load ($= Wbl$) and l the greater dimension of the rectangle. Then the twisting moment on the beam is

$$M_T = \frac{Wl^5}{16(l^4 + b^4)} \qquad (15)$$

It is shown in books on applied mechanics (*see* Perry, p. 358, § 302) that the shearing stress in a rectangular beam subject to twist is approximately

$$f_s = \frac{3M_T(d_1{}^2 + b_1{}^2)}{8d_1{}^3b_1{}^2} \qquad (16)$$

where d is the half-depth and b the half-breadth of the beam.

This should be allowed for in important cases. Thus in the practical case last calculated we have

$W = \frac{1}{10} \times 4 \times 4{\cdot}8 = 1{\cdot}92$ tons, $l = 4{\cdot}8$, and $b = 4{\cdot}0$.

Let b be 2 ft., and $d = 3$ ft. We have

$$M_T = \frac{1{\cdot}92 \times (4{\cdot}8)^5}{16(4{\cdot}8^4 + 4^4)} = 0{\cdot}39 \text{ ft.-ton}$$

$$f_s = \frac{3 \times 0{\cdot}39(3^2 + 2^2)}{8(3^3 \times 2^2)} = 0{\cdot}017 \text{ ton per sq. ft.}$$

The resultant is here seen to be unimportant, but obviously other dimensions might greatly increase its value.

Further it should be noticed that this same moment which holds the edge of the slab is transmitted to the walls as a bending moment, and must be considered in designing the walls, as has already been mentioned in the beginning of this chapter.

Sufficient has been said to show that the key-note to

monolithic construction is continuity of stress. There must of course be a corresponding continuity of construction, and special precautions must be taken to allow for contraction and settlement of the work.

CHAPTER X

REINFORCED CONCRETE

The question of economising weight and material in masonry construction, while at the same time preserving the advantages of high compressive resistance, has led to the practice of reinforcing concrete with steel bars and rods. Numerous patent systems are in vogue, among which particular mention should be made of the Hennebique and Kahn systems.

The essential principle underlying all these systems is that of placing steel where tension is to be resisted. The exact theory is as yet in a very uncertain state, but certain simple rules may be given for designing which will serve in most cases. For more elaborate formulæ the reader is referred to the numerous text-books on the subject.

We will first consider the case of a simple beam with one reinforcing rod (Fig. 20).

It will be remembered that in dealing with all beam problems there is a rule

$$\left.\begin{matrix}\text{Bending}\\ \text{moment}\end{matrix}\right\} = \frac{\text{Edge stress} \times \text{Mom. inertia of section}}{\text{Edge distance from } n \text{ axis}}$$

This rule still applies, but there is a preliminary difficulty in regard to the position of the neutral axis.

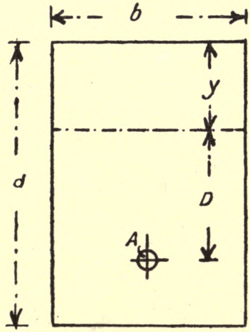

FIG. 20.

This is no longer in the centre of the sectional area, as is the case with all homogeneous materials. There seems to be little doubt as to the fact of the dependence of this quantity on the ratio of the elasticity modulus of concrete to that of steel. This ratio is commonly assumed as 1 : 10 or 1 : 12, but there is considerable difficulty in finding the modulus in the case of concrete. In a case such as is shown the neutral axis is often taken as at one-third the depth from the top edge, *i.e.*, $y = \frac{d}{3}$.

The bending moment is of course easily found when y is known; *i.e.*,

$$M = f_t \mathrm{AD} \times \frac{f_c b y^2}{3} \qquad (1)$$

where f_t is the tensile stress in the steel, f_c the compressive stress in the concrete, and b the width of the section.

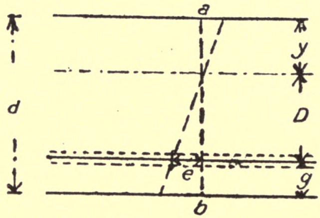

FIG. 21.

This formula assumes that the tensile strength of the concrete is neglected, and that the compressive stress in the concrete varies as the distance from the neutral axis. The above assumption as to the position of the neutral axis is known to be false in many cases, and Hatt, Considine, and others have devised rules to find the true value of y in terms of d.

The following method recently worked out by the author will apply in most simple cases:

Let any section ab of the beam (Fig. 21) be subject to bending and assume that it remains plane. (S. Venant's hypothesis.) Then since e, the strain in the

armature, $= \dfrac{f_t}{E_s}$, where E_s is the modulus for steel, the strain in the concrete at a distance O above the neutral axis (supposing it be possible to go so far) is $\dfrac{f_c D}{y E_c}$,* and these two are equal.

Then
$$\frac{f_t}{E_s} = \frac{f_c \,.\, D}{a E_c} \qquad (2)$$

Also, since bending is resisted by a simple couple, the resultant compressive and tensile forces are equal to one another, and we have

$$_t A = \frac{f_c}{2} \,.\, by \qquad (3)$$

From (3) we have

$$\frac{f_t}{f_c} = \frac{by}{2A},$$

and from (2)

$$\frac{f_t \,.\, y}{f_c D} = \frac{E_s}{E_c},$$ which is taken as 10 : 1 generally.

Eliminating f_t/f_c, we have

$$\frac{by^2}{2AD} + \frac{E_s}{E_c} = 10 \qquad (4)$$

* The stress of the upper edge is f_c and the strain $\dfrac{f_c}{E_c}$. At unit distance from the neutral axis the strain is $\dfrac{f_c}{y E_c}$, and at D from the neutral axis $\dfrac{f_c D}{y E_c}$.

Now $D = d - (y + g)$, which substitute, and we have

$$\frac{by^2}{2A[d - (y + g)]} = 10$$

$$by^2 = 20A[d - (y + g)]$$

$$= 20Ad - 20Ay - 20Ag.$$

Hence

$$by^2 + 20Ay = 20A(d - g)$$

$$y^2 + \frac{20A}{b} \cdot y + \left(\frac{10A}{b}\right)^2 = \frac{20A(d - g)}{b} + \left(\frac{10A}{b}\right)^2$$

$$y = -\frac{10A}{b} \pm \sqrt{\frac{20A(d - g)}{b} + \left(\frac{10A}{b}\right)^2}.$$

Now A is generally expressible as a fraction of bd, the sectional area of the beam, so that we have, taking

$$A = \frac{bd}{n},$$

$$y = -\frac{10bd}{bn} \pm \frac{1}{bn}\sqrt{20nb^2d^2 - 20nb^2dg + 100b^2d^2}$$

$$= -\frac{10d}{n} \pm \frac{d}{n}\sqrt{20n + 100 - \frac{20n}{m}} \qquad (5)$$

where g has been taken as $\frac{d}{m}$.

Thus when $n = 100$ and $m = 6$ (a common proportion), $y = {\cdot}32d$ or $-{\cdot}52d$, the latter being an impossible value. It will be noticed that this agrees with the first assumption, but if $A = \frac{bd}{50}$ then y is ${\cdot}5d$ (nearly).

An alternative form of the bending-moment equation may be obtained by use of the simple couple principle. The centre of pressure in the compression area is $\frac{2}{3}y$ from the neutral axis, so that the moment may be written

$$M = f_t A(D + \tfrac{2}{3}y) = \frac{f_c}{2} \cdot by(D + \tfrac{2}{3}y) \qquad (6)$$

From these rules (*i.e.*, formulæ (1) or (6) and (5)) the dimensions may be readily computed. It is necessary to assume values for n, m, and b, and readjust them if d becomes disproportionate.

Reinforced concrete columns are somewhat difficult to design, since the bending moment to which they may be exposed is quite indeterminate.

A rule frequently employed is

$$W = 400(A_c + 15A_s) \qquad (7)$$

where W is the total load in lbs. and A_c and A_s are the areas respectively of concrete and steel in square inches. This rule, of course, only allows for simple compressive stress (400 lbs. per sq. in. for concrete, and 6000 lbs. per sq. in. for steel). The effect of the concrete on the steel is to slightly lessen its resistance by reason of the lateral constraint produced and initial strains due to contraction. A not unusual practice is to design the column as if of concrete alone, and then add 5 per cent.

of steel (*i.e.*, percentage of sectional area) to provide against possible bending stresses.

In the case of arches and dams we proceed, as in masonry, to find the position of the resultant through the mass and compute the bending moment about the centre of the section by multiplying the normal component of that resultant into the distance between it and the centre of the section. There is no longer any need to restrain this resultant to the middle third, although, of course, the moment will increase rapidly as the resultant becomes more remote. It is usual to reinforce the wall or arch with rods on both sides to take the bending. If we write

$$M = f_t A d \tag{8}$$

where M is the bending moment, A is the total sectional area of the rods on one side, and d their mean distance from the centre line, the reinforcement will be ample.

The following values for stresses, &c., will be useful:

	Chicago Regulations. Lbs. per sq. in.	Galbraith. Lbs.
Concrete (8 : 1)		
Ultimate f_c	2000	—
Extreme fibre stress . .	500	430
Shearing stress . . .	75	20

	Chicago Regulations. Lbs. per sq. in.	Galbraith. Lbs.
Concrete (8 : 1)—*contd.*		
Direct compression (col.?) .	350	700
Adhesion of concrete to steel	75	70 to 570
Steel		
Tensile stress . .	$\frac{1}{3}$ elastic limit	12,800
Shearing stress . . .	10,000	10,000–17,000
Ratio E_s to E_c . . .	12	10

I quote the Chicago figures as typical and useful for calculation. The second column contains figures collected by Mr. A. R. Galbraith, A.M.I.C.E. (Ireland), and published in his paper to the Institute of Civil Engineers (Ireland) in 1904. This paper contains the essential theory and practice of reinforced concrete construction peculiarly well condensed.

In all the cases where bending occurs it is necessary to provide for shearing forces which accompany the flexural stress.

Thus in the case of the beam subject to simple bending there is a shearing force whose maximum intensity occurs between the neutral axis and the reinforcement, being there equal to the total stress in

the latter. Above the neutral axis it diminishes, becoming zero at the top surface of the beam.

To prevent failure in this respect there are various devices. The Hennebique system adopts wrought-iron stirrups which clip round the reinforcement and bed vertically into the concrete. In the Kahn system the reinforcing rods are bent upwards at convenient intervals to an angle of 45°, being thus put in tension to balance the vertical shearing force.

Assuming a parabolic distribution of the shearing stress above the neutral axis, the following expression follows:

$$S = \tfrac{2}{3} f_s by + f_s bD + F_s A \qquad (9)$$

where S is the total shearing force in lbs., f_s is the shearing stress per sq. in. of concrete, and F_s the shearing stress per sq. in. of steel.

The horizontal shearing force is of equal intensity, so that if stirrups or plates be used we may write

$$S = nfa \qquad (10)$$

where S is the total shearing force in lbs. at the section considered, n is the number of plates or stirrups in the immediate neighbourhood of the section, a is the surface of contact between the stirrups and the concrete, and f is the shearing stress in lbs. per sq. in. for concrete on steel.

If Kahn bars or similar diagonal reinforcements be employed, making an angle of 45° with the horizontal,

$$S = mf_t A \div \sqrt{2} \qquad (11)$$

where m is the number of bars occurring at the section considered, f_t is the tensile stress in the steel and A the sectional area.

Up to this point we have treated the question just as we do beams. There are, however, certain points to be considered arising from the monolithic (*i.e.*, continuous) character of reinforced concrete construction.

In the first place, reinforced concrete floors are usually attached at all edges, so that it is necessary to consider the case as one of bending, the beam being fixed all round. If w be the load per foot run of a beam fixed at the ends it will be remembered that the maximum bending moment is $\frac{wl^2}{12}$, while if it be fixed at the edges it generally remains less than $\frac{wl^2}{24}$, *i.e.*, $\frac{w_0 B l^2}{24}$, where w_0 is the weight per super foot and B the breadth of the slab.

Furthermore, in such case there is a reversal of the bending moment at a point midway between the centre and the supports, so that the reinforcement is diagonally passed from the under side of the beam in the central

parts to the upper side at the supports. In quays and some similar cases this is provided for by reinforcing at both top and bottom edges.

Under this latter arrangement it is simplest to regard the bending moment as expressed by the following rule:

$$M = f_t A \Delta \qquad (12)$$

where f is the stress (tensile) per sq. in. of *either* top or bottom reinforcement, A the total area of top *or* bottom

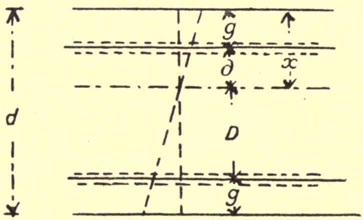

FIG. 22.

reinforcement, and Δ the distance between the two. The concrete in this case simply serves to support the steel, protect it, and reduce deflection. The beam may be regarded as having concrete in compression if necessary, and the neutral axis may be found in much the same manner as before. In the case here considered I assume that the upper and lower reinforcements are equal to one another (this is not essential), and that

they are equidistant from the edges of the beam. By the notion of a simple couple we have

$$f_{cs}\mathrm{A} + \tfrac{1}{2} f_{cc} bx = f_{ts}\mathrm{A} \qquad (13)$$

where f_{cs} is the compressive stress in the steel, f_{ts} the tensile and f_{cc} the compressive stress in the concrete.

By the plane distortion of the sections we have

$$\frac{f_{cc}\mathrm{D}}{x\mathrm{E}_c} = \frac{f_{ts}}{\mathrm{E}_s} \qquad (14a)$$

and

$$\frac{f_{cs}}{\mathrm{E}_s} = \frac{f_{cc}\delta}{x\mathrm{E}_c}. \qquad (14b)$$

Hence we have

$$\frac{f_{cc}\mathrm{D}}{f_{ts}x} = \frac{\mathrm{E}_c}{\mathrm{E}_s} \text{ and } \frac{f_{cs}x}{f_{cc}\delta} = \frac{\mathrm{E}_s}{\mathrm{E}},$$

and from this

$$\frac{f_{cc}}{f_{ts}} = \frac{\mathrm{E}_c x}{\mathrm{E}_s\mathrm{D}} \text{ and } \frac{f_{cs}}{f_{ts}} = \frac{\delta}{\mathrm{D}}.$$

Substituting these values in (13), we arrive at

$$\frac{\mathrm{E}_c x\mathrm{A}}{\mathrm{E}_s\mathrm{D}} + \frac{\delta bx}{2\mathrm{D}} = \mathrm{A} \qquad (15)$$

Writing

$$\mathrm{D} = d - (x + g)$$

and

$$\delta = (x - g)$$

and substituting, we have

$$\frac{E_c x A}{E_s[d - (x + g)]} + \frac{(x - g)bx}{2[d - (x + g)]} = A.$$

Transferring the d in the denominator to the right-hand side and simplifying we get

$$x^2 - \frac{2\left(\frac{bg}{2} + \frac{{}_c . A}{E_s} + A\right)x}{b} = \frac{2A(d - g)}{b} \qquad (16)$$

which, solved as a quadratic, leads to

$$x = \left(\frac{g}{2} + \frac{E_c A}{E_s b} + \frac{A}{b}\right) \pm \sqrt{\frac{2A(d - g)}{b} + \left(\frac{g}{2} + \frac{E_c A}{E_s b} + \frac{A}{b}\right)^2}$$

If we now procee to substitute, as before,

$$A = \frac{bd}{n}, g = \frac{d}{m}, \text{ and } \frac{E_c}{E_s} = \frac{1}{10}.$$

we get x in terms of d:

$$x = \left(\frac{d}{2m} + \frac{d}{10n} + \frac{d}{n}\right) \pm$$

$$\sqrt{\frac{2d\left(d - \frac{d}{m}\right)}{n} + \left(\frac{d}{2m} + \frac{d}{10n} + \frac{d}{n}\right)^2} \qquad (17)$$

If $n = 100$, and $m = 6$, x is nearly $\frac{d}{4}$. As might be expected, the compressive resistance of steel decreases the compressive resistance of the concrete, and the vertical axis is higher.

The moment which the beam is capable of resisting is then

$$f_{ts}A(D + \delta) + \frac{f_{sc}by^2}{3} \qquad (18)$$

[Compare with (12), and note that $\Delta = D + \delta$, the second term being the moment of the concrete stress.]

The assumption which underlies the formulæ given above (that the deformed beam has its originally plane sections still plane) is open to certain objections, and many Continental engineers have suggested alternative and more complex rules. On the score of simplicity, however, I have preferred to employ the above method, and the results do not generally depart much from those obtained by experiment.*

There is now a copious literature on this subject, the more important text-books being:

Twelvetree's *Reinforced Steel Construction.*

Christophe's *Le Béton Armé.* Bérangeur, Paris, 1902.

* Thus Professor Liter assumes a paraboloid distribution of compressive stress in the concrete, so that the stress at any distance z from the neutral axis varies as κz^2. Students will do well to obtain the formulæ for the position of the neutral axis on this assumption and compare with the above.

Morel's *Le Cément Armé et ses Applications*, 1902.
Ritte's "Die Bauweise Hennibique," in *Schweizerische Bauzeitung*, 1899.

Considerable information may also be derived from the handbooks of the Hennebique, Kahn, and other companies.

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